TOPOLOGY PROCEEDINGS
Volume 7, 1982
Pages 279–292

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

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0. Introduction

Every quotient image of a metric space is actually the quotient image of a locally compact metric space [4]. However, every closed image (closed s-image) of a metric space need not be the closed image (closed s-image) of a locally compact metric space. Indeed, any non-locally compact, metric space is not the closed image of a locally compact metric space.

So, the for image Y of a metric space (paracompact M-space) under a closed map, a closed s-map, or a pseudo-open s-map, we give a necessary and sufficient condition for Y to be the image of a locally compact metric space (locally compact paracompact space) under the respective kind of map. Also, we show that every Fréchet space which is the quotient s-image of a locally compact metric space is a Lašnev space which can be decomposed into a discrete closed subspace and a locally compact subspace.

We assume that all spaces are regular; all maps are continuous and onto.

1. Closed Images of Locally Compact Paracompact Spaces

In [9], E. Michael introduced the notion of bi-k-spaces and showed that every bi-k-space is precisely the bi-quotient image of a paracompact M-space. Hence, every paracompact

M-space is bi-k. Recall that a space is a paracompact M-space if it admits a perfect map onto a metric space.

A space X is a *bi-k-space* [9; Lemma 3.E.2], if whenever F is a filter base accumulating at x in X, then there exists a k-sequence (F_n) in X such that $x \in \overline{F \cap F_n}$ for all F_n and all $F \in \mathcal{F}$. Here, a decreasing sequence (F_n) of subsets of X is a k-sequence, if $K = \cap_{n=1}^{\infty} F_n$ is compact in X and every neighborhood of K contains some F_n. We can assume that all F_n are closed in X.

Theorem 1.1. *The following are equivalent.*

(1) Y *is the closed image of a paracompact bi*-k-*space, and each closed (or closed* σ*-compact)* M-*subspace of* Y *is locally compact.*

(2) Y *is the closed image of a locally compact paracompact space.*

Proof. (2) ⇒ (1): Let f be a closed map from a locally compact paracompact space onto Y. Let Y_1 be a closed, M-subspace of Y, and $X_1 = f^{-1}(Y_1)$. Then $g = f|X_1$ is a closed map from a paracompact space X_1 onto an M-space Y_1. Thus, each $\partial g^{-1}(y)$ is compact by [7; Corollary 2.2]. Hence, as in the proof of [7; Corollary 1.2], there is a closed subset F of X_1 such that $g|F$ is a perfect map onto Y_1. Since F is locally compact, so is Y_1. Hence each closed, M-subspace of Y is locally compact.

(1) ⇒ (2): Let $f: X \to Y$ be a closed map with X paracompact bi-k. Let $y \in Y$. Then we will prove that each point of $f^{-1}(y)$ has a neighborhood contained in the inverse image of some compact subset of Y. To see this, suppose

not. Then there is a point $x_0 \in f^{-1}(y)$ such that for any neighborhood V of x_0 and for any compact subset K of Y, $V \not\subset f^{-1}(K)$. Let $\mathcal{F} = \{X - f^{-1}(K); K \text{ is compact in } Y\}$. Then $\mathcal{F}$ is a filter base accumulating at x_0. Since X is a bi-k-space, there is a k-sequence (F_n) in X with $x_0 \in \overline{F_n \cap F}$ for all F_n and all $F \in \mathcal{F}$. Obviously, $(f(F_n))$ is a k-sequence in Y. Suppose that all $f(F_n)$ are not compact. Now, recall the well-known result due to E. Michael: Every closed image of a paracompact space is paracompact (see, [2; Theorem 2.4, p. 165]). Thus Y is paracompact. Then, the $f(F_n)$'s are not countably compact. Hence there are closed, countable discrete subsets D_n of $f(F_n)$. Let $C = \cap_{n=1}^{\infty} f(F_n)$ and $Y_0 = C \cup \cup_{n=1}^{\infty} D_n$. Then Y_0 is closed in Y. Let Z be the quotient space obtained from Y_0 by identifying the compact subset C to a point. Then it is easy to show that Y_0 is the perfect pre-image of a countable metric space Z and that Z is not locally compact. Thus, Y_0 is a closed, σ-compact M-subspace of Y which is not locally compact. This is a contradiction to the hypothesis of Y. Hence some $f(F_{n_0})$ is compact. Let $K_0 = f(F_{n_0})$. Then K_0 is compact in Y. But, $x_0 \in \overline{(X - f^{-1}(K_0)) \cap F_{n_0}} \subset \overline{(X - f^{-1}(K_0)) \cap f^{-1}(K_0)} = \emptyset$. This contradiction implies that each point of $f^{-1}(y)$ has a neighborhood which is contained in the inverse image of some compact subset of Y. Let $\mathcal{V} = \{V; V \text{ is open in } X \text{ with } V \subset f^{-1}(K) \text{ for some compact } K \subset Y\}$. Then $\mathcal{V}$ is an open covering of a paracompact space X. Thus $\mathcal{V}$ has a locally finite closed refinement $\mathcal{C}$. Let $f(\mathcal{C}) = \{f(C): C \in \mathcal{C}\}$. Then, since f is closed, $f(\mathcal{C})$ is a hereditarily closure-preserving cover of compact subsets of Y. Let Z be the topological sum of elements of $f(\mathcal{C})$, and let $g: Z \to Y$ be the obvious map. Then g is a closed map from a locally compact paracompact space Z onto Y.

We shall call a map *Lindelöf* if every point-inverse is a Lindelöf space.

Corollary 1.2. *The following are equivalent.*

(1) Y *is the closed Lindelöf image of a bi-*k*-space, and* Y *is a paracompact space in which every closed (or closed σ-compact)* M*-subspace is locally compact.*

(2) Y *is the closed Lindelöf image of a locally compact paracompact space.*

Proof. It suffices to prove (1) ⇒ (2). Let $f: X \to Y$ be a closed Lindelöf map with X bi-k. Let $y \in Y$. Then, by the proof of (1) ⇒ (2) of the previous theorem, there is a sequence $\{V_n;\ n \in N\}$ of open subsets of X such that $f^{-1}(y) \subset \cup_{n=1}^{\infty} V_n$ and each $f(\overline{V}_n)$ is compact. Since f is closed, $y \in \operatorname{int}(\cup_{n=1}^{\infty} f(\overline{V}_n))$. This shows that Y is locally σ-compact. Since Y is paracompact, there is a locally finite closed cover of σ-compact subspaces F_α. Since f is a closed Lindelöf map, each $f^{-1}(F_\alpha)$ is Lindelöf. Thus each F_α is the closed image of a Lindelöf (hence paracompact) bi-k-space. Thus, by the proof of the previous theorem, each F_α has a countable hereditarily closure-preserving cover of compact subsets. So, each F_α is the closed image of a locally compact, Lindelöf space. Hence Y is the closed Lindelöf image of a locally compact paracompact space. That completes the proof.

Let Y be the closed image of a paracompact space X. Then every compact subset of Y is the image of some compact subset of X [7; Corollary 1.2]. So, in view of the proof of Theorem 1.1 and Corollary 1.2, we have

Corollary 1.3. *Let* Y *be the closed image (resp. closed Lindelöf image) of a paracompact bi*-k-*space* X. *Then* Y *is the closed image (resp. closed Lindelöf image) of the topological sum of some compact subsets of* X *if and only if every closed (σ-compact)* M-*subspace of* Y *is locally compact.*

Now we shall consider Lašnev spaces. Recall that a space is *Lašnev* if it is the closed image of a metric space. Let f: X → Y be a closed map with X metric. Then for each x ∈ X and for each decreasing local base $\{V_n;\ n \in N\}$ at x, $(f(\overline{V}_n))$ is a k-sequence with $\cap_{n=1}^{\infty} f(\overline{V}_n)$ a single point f(x). Also, every compact subset of Y is metrizable. Then, by the proof of Theorem 1.1, we have

Corollary 1.4. *The following are equivalent.*

(1) Y *is the closed image (resp. closed* s-*image) of a metric space, and each closed (or countable closed) metric subspace is locally compact.*

(2) Y *is the closed image (resp. closed* s-*image) of a locally compact, metric space.*

Let X be a space and let $\mathcal{C}$ be a covering (not necessarily closed or open) of X. Then X has the *weak topology* with respect to $\mathcal{C}$ provided that, for A ⊂ X, if A ∩ C is closed in C for all C ∈ $\mathcal{C}$, then A is closed in X. If $\mathcal{C}$ is a closed covering, as a stronger notion, let us recall that X has the *hereditarily weak topology* with respect to $\mathcal{C}$, or equivalently X is *dominated* by $\mathcal{C}$, provided that, for every $\mathcal{C}' \subset \mathcal{C}$, if $A \subset \cup\mathcal{C}'$ and $A \cap C$ is closed in C for all $C \in \mathcal{C}'$, then A is closed in X.

Recall that every space having the weak topology with respect to an increasing, countable closed cover is dominated by the cover. Also, every CW-complex is dominated by the cover of all finite subcomplexes.

Not every space dominated by a countable cover of compact metric subspaces can be decomposed into a σ-discrete subset and a locally compact metric subspace. Indeed, this can be seen by the countable CW-complex obtained from the topological sum of countably many triangles, $\Delta a_i b_i c_i$, by identifying all of segments, $\overline{a_i b_i}$, to a segment. So, as an application of Theorem 1.1, we shall consider the decomposition of spaces having the weak topology with respect to a closed covering of locally compact subspaces.

Recall that a space is *Fréchet* if, whenever $x \in \overline{A}$, then there is a sequence in A converging to the point x.

Proposition 1.5. *For a space* Y *having a closed cover* $\mathcal{F}$ *of locally compact subspaces, we define the following properties.*

(a) Y *is dominated by* $\mathcal{F}$.

(b) Y *has the weak topology with respect to* $\mathcal{F}$ *such that* $\mathcal{F}$ *is point-countable, e.g.,* Y *is the quotient Lindelöf image of a locally compact paracompact space.*

Then Y *is the union of a discrete closed subspace and a locally compact subspace if* Y *satisfies* (1) *or* (2) *below.*

(1) *Either* (a) *or* (b) *holds, and* Y *is the closed image of a paracompact bi*-k-*space.*

(2) Y *is a Fréchet space, and* (b) *holds.*

Proof. Case (1): We will prove that Y is the closed image of a locally compact paracompact space. Hence, by [10; Theorem 4], Y is decomposed as a discrete closed subspace and a locally compact subspace. To prove that, from Theorem 1.1, it suffices to show that every closed, σ-compact M-subspace F of Y is locally compact. If Y satisfies (a), then it is easy to check that every compact subset of Y is contained in a finite union of elements of $\mathcal{F}$. Then F is contained in a countable union of elements $F_i \in \mathcal{F}$. Since F is closed, it is obvious that F has the weak topology with respect to a countable, closed cover $\{F \cap F_i; i = 1,2,\cdots\}$ of locally compact subspaces. This shows that the case (a) reduces to (b). So we assume that Y satisfies (b). It follows from [14; Lemma 6], for each k-sequence (A_n) in Y, that some A_{n_0} is contained in a finite union of elements of $\mathcal{F}$, so A_{n_0} is locally compact. Hence, in view of the proof of Theorem 1.1, Y is the closed image of a locally compact paracompact space.

Case (2): Let $D = \{y \in Y; y \notin \text{int}(\cup\mathcal{F}')$ for any finite $\mathcal{F}' \subset \mathcal{F}\}$. Then it is sufficient to show that D is a discrete closed subset of Y. To see that D is discrete in Y, suppose not. Then there exist a sequence $y_n \in D$ and a point $y_0 \notin D$ with $y_n \to y_0$. Let $K = \{y_n; n \in N\} \cup \{y_0\}$, and

$\{F \in \mathcal{F};\ F \cap K \neq \emptyset\} = \{F_1, F_2, \cdots\}$, and let $T_n = \cup_{i=1}^{n} F_i$. Since $y_n \in D$ (hence, $y_n \in \overline{Y - T_n}$), there exist sequences $S_n = \{y_{nj};\ j = 1,2,\cdots\}$ converging to y_n, and $S_n \cap T_n = \emptyset$. Since $y_0 \in \overline{\cup_{n=1}^{\infty} S_n}$ and $y_n \neq y_0$, there exists a sequence $S = \{y_k';\ k = 1,2,\cdots\}$ converging to y_0 with $y_k' \in S_{n_k}$. But, each convergent sequence together with the limit point (hence, a compact subset) of Y is contained in a finite union of elements of $\mathcal{F}$. Thus, for some n_0 and a subsequence $S_0 = \{y_{k(j)}';\ j = 1,2,\cdots\}$, $S_0 \cup \{y_0\} \subset T_{n_0}$. Thus, for any $n_{k(j)} = m \geq n_0$, $S_m \cap T_m \neq \emptyset$, a contradiction. Hence D is discrete in Y.

2. Fréchet Spaces and Lašnev Spaces

Not every Fréchet space having the weak topology with respect to a point-finite closed cover of metric (hence Lašnev) subspaces is Lašnev. Indeed, let X be the upper half plane. For each real number r and each $n \in N$, let $\{(x,y);\ y = |x - r| < \frac{1}{n}\}$ be a basic neighborhood of $(r,0)$, and let the other points be isolated. Then X is a first countable (hence Fréchet) space having the weak topology with respect to a point-finite clopen cover of metric subspaces. But X is not Lašnev, for it is not normal. So, in terms of weak topology, we shall consider conditions that imply every Fréchet space with Lašnev pieces is Lašnev.

Proposition 2.1. *Let* X *be a Fréchet space having the weak topology with respect to a closed cover* $\mathcal{F}$ *of Lašnev subspaces. If* (1) *or* (2) *below holds, then* X *is Lašnev.*

(1) *$\mathcal{F}$ is countable.*

(2) *$\mathcal{F}$ is point-countable and each element of $\mathcal{F}$ is a separable space.*

Proof. Case (1): Let $\mathcal{F} = \{F_n; n \in N\}$, $C_n = \cup_{i=1}^n F_i$, and let $M_n = \overline{C_n - C_{n-1}}$, $C_0 = \emptyset$. Let M be the topological sum of $M_n (n \in N)$, and let $f: M \to X$ be the obvious map. Since X is Fréchet space having the weak topology with respect to $\{C_n; n \in N\}$ with $C_n \subset C_{n+1}$, by the proof of F. Siwiec [12; Proposition 2(a)], we show that f is a closed map without any topological property of F_n. Since each F_n is now Lašnev, so is M. Then X is Lašnev.

Case (2): Every separable Lašnev space is obviously the closed image of a separable metric space. Thus, each element of $\mathcal{F}$ is an $\aleph_0$-space [8], that is, a space with a countable k-network. Thus X has the weak topology with respect to a point-countable cover of $\aleph_0$-subspaces. Since X is Fréchet, by [5; Corollary 8.9], X is the topological sum of $\aleph_0$-spaces. Hence, it is sufficient that every closed, $\aleph_0$-subspace S of X is Lašnev. To see this, let $\mathcal{N}$ be a countable k-network for S, that is, whenever $K \subset U$ with K compact and U open in S, then $K \subset \cup\mathcal{N}' \subset U$ for some finite $\mathcal{N}' \subset \mathcal{N}$. We assume that each element of $\mathcal{N}$ is closed in S, and that $\mathcal{N}$ is closed under finite unions and intersections. Now, let K be any compact subset of S, and each $N_n \in \mathcal{N}$ contain the set K, and let $K_n = \cup_{i=1}^n N_i$ for each n. Then $K_n \in \mathcal{N}$ and (K_n) is a k-sequence in S with $K = \cap_{n=i}^\infty N_n$. On the other hand, the closed subset S of X has the weak topology with respect to a point-countable closed cover

$\mathcal{C} = \{S \cap F;\ F \in \mathcal{F}\}$. Hence by [14; Lemma 6], some K_{n_0} is contained in a finite union of elements of $\mathcal{C}$. Since each element of $\mathcal{C}$ is Lašnev, so is K_{n_0}. This implies that $\mathcal{N}' = \{N \in \mathcal{N};\ N$ is Lašnev$\}$ is a k-network for S. Thus, S has the weak topology with respect to the countable closed cover $\mathcal{N}'$, because S is Fréchet (hence a k-space) and each compact subset of S is contained in an element of $\mathcal{N}'$. Hence, by (1), S is a Lašnev space. That completes the proof.

F. Siwiec [12] showed that closed images of locally compact, separable metric spaces are precisely the hemicompact Fréchet spaces in which every compact subset is metrizable. As for closed s-images of locally compact metric spaces, we have the following characterization and a relationship between closed s-images and pseudo-open s-images. Recall that a map $f\colon X \to Y$ is *pseudo-open* [1] (i.e., *hereditarily quotient*) if for any $y \in Y$ and for any neighborhood U of $f^{-1}(y)$, $y \in \operatorname{int} f(U)$.

Theorem 2.2. *The following are equivalent.*

(1) Y *is the pseudo-open* s-*image of a metric space, and each closed (or countable closed) metric subspace is locally compact.*

(2) Y *is the pseudo-open* s-*image of a locally compact, metric space.*

(3) Y *is the closed* s-*image of a locally compact, metric space.*

(4) Y *is a Fréchet space having the weak topology with respect to a point-countable cover of compact metric subspaces.*

Proof. Since every closed map is pseudo-open, (3) ⇒ (2) is obvious.

(2) ⇒ (1): Let $f: X \to Y$ be a quotient s-map with X a locally compact, metric space. Let F be a closed, metric subspace of Y. Then $g = f|f^{-1}(F)$ is a quotient s-map from a locally compact space $f^{-1}(F)$ onto a metric space F. Thus, F is locally compact by [9; Propositions 3.3(d) and 3.4(a)]. Thus every closed metric subspace of Y is locally compact.

(4) ⇒ (3): From Proposition 2.1, Y is Lašnev. Thus, as in the proof of Proposition 1.5, Y is the closed image of a locally compact, metric space (under f). Since Y has the weak topology with respect to a point-countable cover of compact subsets, every $\partial f^{-1}(y)$ is Lindelöf by [13; Remark 4]. Thus Y is the closed s-image of a locally compact, metric space.

(1) ⇒ (4): Let $f: X \to Y$ be a pseudo-open s-map with X metric. Since each closed metric subspace of Y is locally compact, as in the proof of Theorem 1.1 or Corollary 1.4, X has a locally finite closed cover $\mathcal{F}$ each of whose element is contained in the inverse-image of a compact subset of Y. Since $\mathcal{F}$ is a locally finite closed cover of X and f is a quotient s-map, Y has the weak topology with respect to a point-countable cover $f(\mathcal{F})$ each of whose closure is compact, hence separable metric by [3; Corollary 3]. While, Y is Fréchet, for every pseudo-open image of a metric space is Fréchet [1]. Thus, by [5; Corollary 8.9], Y is the topological sum of $\aleph_0$-spaces. But Y has the weak topology with respect to a point-countable cover each of whose closure is

compact. Hence, by the same way as in the proof of Proposition 2.1, Y has the weak topology with respect to a point-countable cover of compact metric subspaces. That completes the proof.

Since every quotient map onto a Fréchet space is pseudo-open [1], using a decomposition theorem [10; Theorem 4] of closed images of locally compact spaces, we have

Corollary 2.3. *Every Fréchet space which is the quotient* s-*image of a locally compact metric space is a Lašnev space which can be decomposed into a discrete closed subspace and a locally compact metric subspace.*

We remark that not every Fréchet space which is the quotient s-image of a metric space is either Lašnev or is the union of a discrete closed subspace and a metric subspace; see [5; Example 9.4], which is Lindelöf, non-separable, and has a point-countable base (hence, it is the open s-image of a metric space by [11]). As for decompositions of Lašnev spaces, N. Lašnev [6] showed that not every countable Lašnev (hence, $\aleph_0$) space is the union of a discrete closed subspace and a metric subspace, but every Lašnev space is at least decomposed as a σ-discrete subset and a metric subspace.

Thus, in view of Corollary 2.3 and Proposition 1.5 (case (2)), we pose the following problem concerning Fréchet spaces.

Problem 2.4. Is every Fréchet space which is the quotient image of a separable metric space (i.e., Fréchet

$\aleph_0$-space) Lašnev, or at least decomposed as a σ-discrete subset and a metric subspace? How about every Fréchet space dominated by a cover of compact metric subspaces (e.g., Fréchet CW-complex)?

If the former is affirmative, then every Fréchet space which is the quotient s-image of a locally separable metric space is Lašnev, or at least decomposed as a σ-discrete subset and a metric subspace, because such spaces are characterized as the Fréchet spaces which are the topological sum of $\aleph_0$-spaces by [5; Proposition 8.8] and [8; Corollary 11.5].

As for the latter, if every Fréchet space dominated by a cover of compact metric subspaces is Lašnev, then such a space can be decomposed into a discrete closed subspace and a locally compact metric subspace by Proposition 1.5 (case (1)).

References

1. A. Arhangel'skii, *Some types of factor mappings, and the relations between classes of topological spaces*, Soviet Math. Dokl. 4 (1963), 1726-1729.
2. J. Dugundji, *Topology*, Allyn and Bacon, Boston.
3. V. V. Filippov, *Quotient spaces and multiplicity of a base*, Math. USSR-sb. 9 (1969), 487-496.
4. S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. 57 (1965), 107-115.
5. G. Gruenhage, E. Michael and Y. Tanaka, *Spaces determined by point-countable covers* (to appear).
6. N. Lašnev, *Continuous decompositions and closed mappings of metric spaces*, Soviet Math. Dokl. 6 (1965), 1504-1506.

7. E. Michael, *A note on closed maps and compact sets*, Israel J. of Math. 2 (1964), 173-176.

8. ________, *$\aleph_0$-spaces*, J. Math. Mech. 15 (1966), 983-1002.

9. ________, *A quintuple quotient quest*, General Topology and Appl. 2 (1972), 91-138.

10. K. Morita, *On closed mappings*, Proc. Japan Acad. 32 (1956), 539-543.

11. V. I. Ponomarev, *Axioms of countability and continuous mappings*, Bull. Pol. Acad. Nauk. VIII (1960), 127-134.

12. F. Siwiec, *A note on identifications of metric spaces*, Proc. Amer. Math. Soc. 57 (1976), 340-344.

13. Y. Tanaka, *Closed maps on metric spaces*, Topology and Appl. 11 (1980), 87-92.

14. ________, *Point-countable* k-*systems and product of* k-*spaces*, Pacific J. of Math. 101 (1982), 199-208.

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