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1. Background

All spaces are assumed to be completely regular and Hausdorff.

In 1951, Dowker [3] gave an internal characterization of spaces whose product with the closed unit interval is normal. He showed that $X \times [0,1]$ is normal if and only if X is normal and satisfies a countable form of the property of paracompactness which had been introduced a few years before (countable paracompactness). In 1959, Horne [5] considered when a space has the property that every locally bounded function may be bounded by a continuous function (a cb-space). He showed that cb-spaces are countably paracompact and that the converse holds for normal spaces.

In 1969, Mack [6] gave a characterization of countably paracompact spaces in the fashion of Dowker's characterization of countably paracompact normal spaces. He showed that X is countably paracompact if and only if $X \times [0,1]$ has the following "weak normality property": δ-normal (every closed set which is the intersection of the closures of countably many open sets containing it (regular G_δ-set) can be separated (with disjoint open sets) from any closed set which does not intersect it).

In 1967, Zenor [7] has considered another weak normality property: δ-normally separated (every zero set can be separated (with a Urysohn function) from any closed set which does not intersect it). Mack unified the work of Zenor and Horne by showing that X is a cb-space if and only if $X \times [0,1]$ is δ-normally separated. Thus in summary:

$$\begin{tikzcd}
\text{X normal and ctbly. para.} \arrow[r] \arrow[d, leftrightarrow] & \text{X cb-space} \arrow[r] \arrow[d, leftrightarrow] & \text{X ctbly. para.} \arrow[d, leftrightarrow] \\
\text{X} \times [0,1] \text{ normal} & \text{X} \times [0,1]\ \delta\text{-normally separated} & \text{X} \times [0,1]\ \delta\text{-normal}
\end{tikzcd}$$

The natural conjecture with respect to the weak normality properties is the following:

$$\begin{tikzcd}
\text{X normal} \arrow[r] & \text{X } \delta\text{-normally separated} \arrow[r] & \text{X } \delta\text{-normal}
\end{tikzcd}$$

The first implication is true but in 1969 Mack asked: Is every δ-normally separated space, δ-normal? This question was asked again by Alo and Shapiro in their monograph [1]. As a partial result, Hardy and Juhasz [4] described a weakly δ-normally separated (for definition see p. 254 of [1]) space which is not δ-normal. We construct a δ-normally separated space which is not δ-normal.

2. The Counterexample

We construct a completely regular Hausdorff space X such that:

(X1) Any zero set either (a) is contained in a clopen set which is a normal subspace of X or (b) contains a clopen set whose complement is a normal subspace of X.

(X2) There is a nonempty closed set A, a family of open sets $\{U_i: i \in N\}$ such that $A = \cap\{U_i: i \in N\}$ and for each $i \in N$, $\overline{U}_i \subset U_{i+1}$ and a nonempty closed set B such that $A \cap B = \phi$ for which there do not exist disjoint open sets

U, V such that $U \supset A$ and $V \supset B$.

The idea behind X is as follows: There are regular spaces X with $a, b \in X$ such that any continuous function $f: X \to R$ is such that $f(a) = f(b)$. These spaces are not completely regular. A copy of ω_1 behaves like a point with respect to continuous real-valued functions. That is, if X is a space and $A \subset X$ is a copy of ω_1, then, for any continuous function $f: X \to R$, there is a real number r such that all but countably many $a \in A$ are such that $f(a) = r$. We say $f(A) = r$. There are completely regular spaces X with disjoint copies of ω_1, $A, B \subset X$ such that any continuous function $f: X \to R$ is such that $f(A) = f(B)$. Let us say A and B are tied. We describe the structure of X: There is ω_1 and a sequence of copies of ω_1. Each consecutive pair of copies is tied and ω_1 and each copy are tied but we allow ω_1 to dissociate itself from any finitely many copies at one time.

Lemma 1. *X is a topological space. Let* $X = \omega_1 \cup (\omega_1^2 \times \omega)$. *Topologize X as follows:*

(0) *The* β, γ, nth *nhood of* $\alpha \in \omega_1$ (where $\gamma < \alpha < \beta < \omega_1$) *is* $\{\delta: \gamma < \delta \leq \alpha\} \cup \{(\delta, \xi, m): \gamma < \xi \leq \alpha$ *and* $\beta < \delta < \omega_1$ *and* $m \geq n\}$

(1) *The* β, γth *nhood of* (α, α, n) (where $\gamma < \alpha < \beta < \omega_1$) *is* $\{(\delta, \xi, n): \gamma < \delta \leq \alpha$ *and* $\gamma < \xi \leq \alpha\} \cup \{(\delta, \xi, n-1): \beta < \xi < \omega_1$ *and* $\gamma < \delta \leq \alpha\}$ *where the second summand is ignored if* $n = 0$.

(2) (α, β, n) *for* $\alpha \neq \beta$ *is an isolated point.*

Proof. Any nhood of type (0) is disjoint from

$\{(\alpha,\alpha,n)\colon \alpha < \omega_1$ and $n \in N\}$, any nhood of type (1) is disjoint from ω_1 and any basic open nhood of any (α,α,n) is disjoint from $\{(\beta,\beta,m)\colon \beta < \omega_1$ and $m \neq n\}$. If the intersection of the β,γ,nth nhood of α and the β',γ',n'th nhood of α' contains α'', then it also contains the $\max\{\beta,\beta'\}$, $\max\{\gamma,\gamma'\}$,$\max\{n,n'\}$th nhood of α''. If the intersection of the β,γth nhood of (α,α,n) and the β',γ'th nhood of (α',α',m) contains (α'',α'',p), then $n = p = m$ and it contains the $\max\{\beta,\beta'\}$,$\max\{\gamma,\gamma'\}$th nhood of (α'',α'',p).

Lemma 2. X *is a completely regular space.*

Proof. We show that X is Hausdorff and 0-dimensional. Each point is the intersection of its basic open nhoods so it suffices to show that each element of the base is clopen. The subspace topologies on ω_1 and $\{(\alpha,\alpha,n)\colon \alpha < \omega_1\}$ for each $n \in N$ are the usual ones. Two nhoods of type (0) intersect if and only if they intersect on ω_1. Any basic open nhood of a point in ω_1 with first parameter β is disjoint both from any basic open nhood of any (α,α,n) where $\alpha \leq \beta$ and from the $0,\beta$th nhood of any (α,α,n) where $\alpha > \beta$. Therefore basic open sets of type (0) are clopen. If a basic open nhood of (α,α,n) and a basic open nhood of (α',α',n') intersect then n and n' differ by at most one. If $n = n'$ they intersect on $\{(\beta,\beta,n)\colon \beta \in \omega_1\}$. Any basic open set of type (0) with first parameter α is disjoint from any nhood of (α,α,n), any basic open nhood of (α,α,n) with first parameter β is disjoint from any nhood of $(\beta,\beta,n-1)$ and any basic open nhood of $(\alpha,\alpha,n-1)$ (with second parameter β if $\beta < \alpha$) is disjoint from any nhood of

(β,β,n). Therefore basic open sets of type (1) are clopen.

Lemma 3. X *satisfies* (X1).

Proof. For each $n \in \omega$, $\{(\alpha,\alpha,n): \alpha \in \omega\}$ is homeomorphic to ω_1 and so there exists $\alpha_n \in \omega_1$ and a real number r_n such that, whenever $\alpha > \alpha_n$, $f((\alpha,\alpha,n)) = r_n$. For each $n,k \in \omega$, there exists $\alpha_n^k \in \omega_1$ such that, whenever δ and γ are greater than α_n^k, $|f((\delta,\gamma,n)) - r_n| < \frac{1}{k}$ (otherwise, define inductively $\{\delta_i: i \in \omega\}$ and $\{\gamma_i: i \in \omega\}$ so that, for each $i \in \omega$, $|f((\delta_i,\gamma_i,n)) - r_n| \geq \frac{1}{k}$, $\inf\{\delta_{i+1},\gamma_{i+1}\} > \sup\{\delta_i,\gamma_i\}$ and $\inf\{\delta_0,\gamma_0\} > \alpha_n$; let $\eta = \sup\{\delta_i: i \in \omega\} = \sup\{\gamma_i: i \in \omega\}$ and get a contradiction since $\{(\delta_i,\gamma_i,n): i \in \omega\}$ converges to (η,η,n) and, since $\eta > \alpha_n$, $f(\eta,\eta,n) = r_n$). Let $\alpha^* = \sup\{\alpha_n^k: k,n \in \omega\}$. Whenever δ and γ are greater than α^*, $f((\delta,\gamma,n)) = r_n$. For each $n > 0$ and $\alpha > \alpha^*$, (α,α,n) is in the closure of $\{(\delta,\gamma,n-1): \alpha^* < \delta < \gamma < \omega_1$. Whenever $\alpha^* < \delta < \gamma < \omega_1$, $f((\delta,\gamma,n-1)) = r_{n-1}$ and whenever $\alpha > \alpha^*$, $f((\alpha,\alpha,n)) = r_n$. This implies that, for each $n > 0$, $r_n = r_{n-1}$ and so that there is a real number c such that, for each $n \in \omega$, $r_n = c$. For each $\alpha > \alpha^*$, α is in the closure of $\{(\alpha,\beta,n): \alpha^* < \beta < \alpha < \omega_1$ and so $f(\alpha) = c$. Let $R = \{\alpha: \alpha > \alpha^*\}$ $\{(\alpha,\beta,n): \alpha$ and β are greater than α^* and $n \in \omega\}$.

For each $x \in R$, $f(x) = c$. Claim R is a clopen subset of X. R is open since any basic open nhood of $\alpha \in R$ with second parameter α^* is contained in R and any basic open nhood of $(\alpha,\alpha,n) \in R$ with second parameter α^* is contained in R. R is closed since any basic open neighborhood of a point not in R is disjoint from R. X - R has countably many nonisolated points. Any such space is paracompact and, thus, normal.

Lemma 4. X *satisfies* (X2).

Proof. Let $A = \omega_1$ and, for each $n \in \omega$, let $U_n = \omega_1$ $\{(\alpha,\beta,m): m > n \text{ or } \alpha < \beta;\ \alpha,\beta \in \omega_1,\ m \in \omega\}$. Let $B = \{(\alpha,\alpha,m): \alpha \in \omega_1,\ m \in \omega\}$. Suppose that there exist disjoint open sets U,V such that $U \supset A$ and $V \supset B$. For each $n \in \omega$, $\alpha \in \omega_1$, let $f_n(\alpha) < \alpha$ be such that, for some $\beta \in \omega_1$, the $\beta, f_n(\alpha)$th neighborhood of (α,α,n) is contained in V. Each f_n is a regressive function on ω_1 and so there is an uncountable set A_n contained in ω_1 and $\lambda_n \in \omega_1$ such that, for each $\alpha \in A_n$, $f_n(\alpha) = \lambda_n$. Let $\lambda > \sup\{\lambda_n: n \in \omega\}$. Some β,γ,nth neighborhood of λ is contained in U. Let $\delta \in A_n$ be such that $\delta > \beta$ and $\delta > \lambda$. $(\delta,\lambda,n) \in U \cap V$ and that is a contradiction.

Lemma 5. X *is a* δ*-normally separated space which is not* δ*-normal.*

Proof. Any completely regular space X satisfying (X1) is δ-normally separated. If Z is a zero set in X, there is a decomposition of $X = X_1 \oplus X_2$ such that X_1 is normal and either X_2 is contained in Z or X_2 is disjoint from Z. If A is a closed set disjoint from Z, then at most one of A,Z intersect X_2. To construct a Urysohn function separating Z and A, it suffices to do so in each of X_1 and X_2. In X_2 we may take either the constant 0 function if X_2 is disjoint from A or the constant 1 function if X_2

is disjoint from Z. In X_1, we use the normality of X_1. Any completely regular space satisfying (X2) is not δ-normal.

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