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FACTORWISE RIGIDITY OF THE PRODUCT OF TWO PSEUDO-ARCS

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Introduction

I denotes the interval [0,1]. A *continuum* is a compact connected metric space. If X and Y are continua, an ε-map $f: X \to Y$ is a continuous surjective mapping such that for each $p \in Y$, $f^{-1}(p)$ has diameter less than ε. A continuum X is *arclike* or *chainable* if and only if, for each $\varepsilon > 0$, there exists an ε-map $f: X \to I$. A continuum X is *indecomposable* if and only if it is not the union of two of its proper subcontinua. This is equivalent to each of its proper subcontinua having empty interior. X is *hereditarily indecomposable* if and only if each of its subcontinua is indecomposable. A chainable hereditarily indecomposable continuum is called a *pseudo-arc*. Some background information on pseudo-arcs can be found in [1], [2], [3], [4], [6], and [9]. It is clear that every nondegenerate subcontinuum of a pseudo-arc is a pseudo-arc, and R. H. Bing has shown that all pseudo-arcs are homeomorphic, although we shall not need this fact.

If X and Y are continua then $\pi_1: X \times Y \to X$ and $\pi_2: X \times Y \to Y$ will always denote the projection maps. If d_1 and d_2 are the given metrics on X and Y, the metric d on

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[2]This paper was written when the second author was visiting the University of Delaware.

$X \times Y$ will always be taken as the maximum metric; $d((x,y),(u,v)) = \max\{d_1(x,u), d_2(y,v)\}$. Thus, if $f\colon X \to X_0$ and $g\colon Y \to Y_0$ are ε-maps, then $f \times g\colon X \times Y \to X_0 \times Y_0$ is an ε-map also.

A product of mutually homeomorphic continua is *factorwise rigid* if and only if each homeomorphism of the product is a product of homeomorphisms of the separate factors, followed by a permutation of the factors. Thus, a product $X \times X$ is factorwise rigid if every homeomorphism $H\colon X \times X \to X \times X$ is of the form $H(x,y) = (h(x),k(y))$ or $H(x,y) = (h(y),k(x))$ where h and k are self-homeomorphisms of X. Wayne Lewis [7, problem 60] has asked whether every product of pseudo-arcs is factorwise rigid. This article gives an affirmative answer for a product of two pseudo-arcs. It is known that every product of Menger or Sierpinski Universal curves is factorwise rigid [5] and [11].

If X is a continuum and $A \subseteq X$, $T(A)$ is defined to be the complement of the set of points of X which have a continuum neighborhood missing A. A substantial body of litera ture exists on this set function and related topics. Rather than attempt to give a comprehensive list of references, we refer the interested reader to the articles in Sections II and V of [8], and the bibliographies thereof.

Lemma 1. *Suppose* W *and* M *are subcontinua of* $I \times I$ *and that* $\pi_1(M) = I$ *while* $\pi_2(W) = I$. *Then* $W \cap M \neq 0$.

Proof. This is an immediate consequence of Theorem 28 of [10, p. 156].

Lemma 2. *Suppose* X *and* Y *are chainable continua and* M, W $\subseteq$ X $\times$ Y *are continua such that* $\pi_1(M) = X$ *and* $\pi_2(W) = Y$. *Then* $M \cap W \neq \emptyset$.

Proof. Suppose not. Choose $\varepsilon > 0$ such that $\varepsilon < d(M,W)$, and let $f: X \to I$ and $g: Y \to I$ be ε-maps. Then $f \times g: X \times Y \to I \times I$ is an ε-map also, so that $(f\times g)(M) \cap (f\times g)(W) = \emptyset$, since a point in this intersection would have inverse image under $f \times g$ with diameter greater than ε.

However, $\pi_1((f\times g)(M)) = f(\pi_1(M)) = I$ and $\pi_2((f\times g)(W)) = g(\pi_2(W)) = I$, so that by Lemma 1, $(f\times g)(M) \cap (f\times g)(W) \neq \emptyset$, a contradiction.

Corollary 3. *Let* X *and* Y *be chainable continua and suppose* W *and* M *are subcontinua of* X $\times$ Y *such that* $\pi_1(W) \subseteq \pi_1(M)$ *while* $\pi_2(M) \subseteq \pi_2(W)$. *Then* $W \cap M \neq \emptyset$.

Proof. Subcontinua of chainable continua are chainable (or degenerate) so Lemma 2 applies to $\pi_1(M) \times \pi_2(W)$.

Lemma 4. *If* X *and* Y *are continua and* A *is a closed subset of* X $\times$ Y, *and* $T(A) = X \times Y$, *then either* $\pi_1(A) = X$ *or* $\pi_2(A) = Y$.

Proof. This is a classical argument due to F. B. Jones. Suppose A is closed in $X \times Y$, $\pi_1(A) \neq X$, and $\pi_2(A) \neq Y$. Then there exists a nonempty open subset U of X with $\overline{U} \cap \pi_1(A) = \emptyset$, and there is a point $y \in Y - \pi_2(A)$. Then $(\overline{U}\times Y) \cup (X\times\{y\})$ is a continuum in $X \times Y$ with nonvoid interior missing A so that $T(A) \neq X \times Y$.

Lemma 5. *If* X *and* Y *are continua and* X *is indecomposable, then for any* $a \in X$, *in* $X \times Y$, $T(\{a\}\times Y) = X \times Y$.

Proof. If $W \subseteq X \times Y$ is a continuum with nonvoid interior, $\pi_1(W) = X$, since $\pi_1(W)$ must be a subcontinuum of X with interior. Thus, $a \in \pi_1(W)$ so that $W \cap (\{a\}\times Y) \neq \emptyset$.

Lemma 6. *Suppose* X *and* Y *are indecomposable continua and* $a \in X$ *and* $h: X \times Y \to X \times Y$ *is a homeomorphism. Then either* $\pi_1(h(\{a\}\times Y)) = X$ *or* $\pi_2(h(\{a\}\times Y)) = Y$.

Proof. For any set A and any homeomorphism h, $T(h(A)) = h(T(A))$. Apply Lemmas 4 and 5.

In what follows, P denotes a pseudo-arc.

Lemma 7. *Let* $h: P \times P \to P \times P$ *be any homeomorphism. Then either* $\pi_1(h(\{p\}\times P)) = P$ *for every* $p \in P$, *or* $\pi_2(h(\{p\}\times P)) = P$ *for every* $p \in P$.

Proof. For $i = 1,2$, let $M_i = \{p \in P | \pi_i(h(\{p\}\times P)) = P\}$; by Lemma 6, $M_1 \cup M_2 = P$. Suppose $M_1 \neq P$ and $M_2 \neq P$. Then M_1 and M_2 are closed sets, and so there exist two distinct points $a, b \in M_1 \cap M_2$, since P has no separating point. Then, $\pi_1(h(\{a\}\times P)) = \pi_2(h(\{a\}\times P)) = \pi_1(h(\{b\}\times P)) = \pi_2(h(\{b\}\times P)) = P$. Thus, $h(\{a\}\times P)$ and $h(\{b\}\times P)$ satisfy the hypotheses of Lemma 2, so that $h(\{a\}\times P) \cap h(\{b\}\times P) \neq \emptyset$. But then $(\{a\}\times P) \cap (\{b\}\times P) \neq \emptyset$, so that $a = b$, a contradiction. Hence, either $M_1 = P$ or $M_2 = P$, as claimed.

Let $\theta: P \times P \to P \times P$ be the homeomorphism which interchanges the factors.

Theorem. Let $H: P \times P \to P \times P$ *be a homeomorphism. Then there exist homeomorphisms* $h,k: P \to P$ *such that either* $H = (h\times k)$ *or* $H = \theta\circ(h\times k)$; *in other words* $P \times P$ *is factorwise rigid.*

Proof. By Lemma 7, either $\pi_2(H(\{p\}\times P)) = P$ for every $p \in P$ or $\pi_1(H(\{p\}\times P)) = P$ for every $p \in P$. Assume the latter; the other case is similar. Then $\pi_2(\theta\circ H(\{p\}\times P)) = P$ for every $p \in P$. Suppose for some $p \in P$, $\pi_1(\theta\circ H(\{p\}\times P))$ is nondegenerate. Let $\langle W_n\rangle_{n=1}^{\infty}$ be a decreasing sequence of nondegenerate subcontinua of P whose intersection is $\{p\}$, and let $a \in P$. Then $\langle W_n\times\{a\}\rangle_{n=1}^{\infty}$ is a sequence of continua in $P \times P$ whose intersection is $\{(p,a)\}$, and so $\langle(\theta\circ H(W_n\times\{a\}))\rangle_{n=1}^{\infty}$ has intersection equal to $\{\theta\circ H(p,a)\}$, and $\langle\pi_1(\theta\circ H(W_n\times\{a\}))\rangle_{n=1}^{\infty}$ is a decreasing sequence of continua in P whose intersection is $\{\pi_1(\theta\circ H(p,a))\}$. In particular, $\pi_1(\theta(H(p,a))) \in \pi_1(\theta(H(W_n\times\{a\})))$ for every n, so that for every n, $\pi_1(\theta\circ H(W_n\times\{a\})) \cap \pi_1(\theta\circ H(\{p\}\times P))$ is nonempty, and so for some n, $\pi_1(\theta\circ H(W_n\times\{a\})) \subseteq \pi_1(\theta\circ H(\{p\}\times P))$, since P is hereditarily indecomposable and $\pi_1(\theta\circ H(\{p\}\times P))$ is nondegenerate and so cannot be a subset of $\pi_1(\theta\circ H(W_m\times\{a\}))$ for every m. Let $q \in W_n$ with $q \neq p$. Then $\pi_1(\theta\circ H(\{q\}\times P)) \cap \pi_1(\theta\circ H(\{p\}\times P)) \neq \emptyset$, so that one of these continua is a subset of the other. Without loss of generality, assume

$$\pi_1(\theta\circ H(\{q\}\times P)) \subseteq \pi_1(\theta\circ H(\{p\}\times P)).$$

But $\pi_2(\theta\circ H(\{p\}\times P)) \subseteq \pi_2(\theta\circ H(\{q\}\times P))$ (since both of these are equal to P). Thus, by Corollary 3,

$$\theta\circ H(\{q\}\times P) \cap \theta\circ H(\{p\}\times P) \neq \emptyset,$$

and consquently $(\{q\}\times P) \cap (\{p\}\times P) \neq \emptyset$, since $\theta\circ H$ is 1 to 1. This is a contradiction since $q \neq p$. Therefore, for every $p \in P$, $\pi_1(\theta\circ H(\{p\}\times P)) = \{x\}$ for some $x \in P$. Define $h: P \to P$ by $h(p) = \pi_1\circ\theta\circ H\circ\pi_1^{-1}(p)$, which is well-defined by the above argument. A parallel argument will prove that k: P → P defined by $k(q) = \pi_2 \circ \theta \circ H \circ \pi_2^{-1}(q)$ well-defined. Then, for any $(p,q) \in P \times P$, $\theta \circ H(p,q) = (h(p),k(q))$, or $\theta \circ H = (h \times k)$, so that $H = \theta \circ (h \times k)$. (The other case, that $\pi_2(H(\{p\} \times P)) = P$ for every $p \in P$, yields $H = h \times k$ for some h and k.)

The idea for this paper grew in part out of a conversation between Howard Cook and the first author in the Spring of 1980. At that time, it was proven that there is no homeomorphism of the product P × P which carries the diagonal to a fiber {a} × P. This fact now follows as a corollary to the Theorem in this paper. This is a curious fact, since if X is either a topological group or an n-cell for $n \geq 1$, there is a homeomorphism of X × X carrying the diagonal to any {a} × X, but though the pseudo-arc is homogeneous, and of trivial shape, no such homeomorphism exists.

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