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by

G. Gruenhage and E. Michael

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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1. Introduction

The principal purpose of this note is to prove the following result.

Theorem 1.1. *Every cover* $\mathcal{V}$ *of a regular space* X *by open subsets with Lindelöf boundaries can be shrunk.*

We say that an open cover $\mathcal{V}$ of X can be *shrunk* if there exists an indexed closed cover $\{A_V: V \in \mathcal{V}\}$ of X such that $A_V \subset V$ for every $V \in \mathcal{V}$.[1] Such a cover will be called a *shrinking* of $\mathcal{V}$.

Section 2 is devoted to some preliminary lemmas which may be of independent interest. Section 3 gives the proof of Theorem 1.1, and Section 4 establishes a corollary to Theorem 1.1 which will be applied in [2]. Section 5 offers some examples, and Section 6 raises a related question.

2. Two Lemmas

Call a collection $\mathcal{U}$ of subsets of X *enveloping* if $\overline{U} \subset \cup\mathcal{U}$ for every $U \in \mathcal{U}$.

Lemma 2.1. *If* $\mathcal{V}$ *is a cover of a space* X *by subsets with Lindelöf boundaries, then every* $V \in \mathcal{V}$ *belongs to a countable, enveloping* $\mathcal{U} \subset \mathcal{V}$.

[1]Note that we do not make the stronger requirement that there exists an indexed open cover $\{U_V: V \in \mathcal{V}\}$ of X such that $\overline{U}_V \subset V$ for every $V \in \mathcal{V}$. While the two requirements are clearly equivalent when X is normal (or, more generally, when $\overline{V}$ is normal for every $V \in \mathcal{V}$), they are not equivalent in Theorem 1.1; see Example 5.2.

Proof. Inductively, choose countable subcollections $\mathcal{U}_1, \mathcal{U}_2, \cdots$ of $\mathcal{V}$ such that $\mathcal{U}_1 = \{V\}$ and $\overline{U} \subset \cup\mathcal{U}_{n+1}$ for every $U \in \mathcal{U}_n$. Let $\mathcal{U} = \cup_{n=1}^{\infty}\mathcal{U}_n$. This $\mathcal{U}$ has the required properties.

Lemma 2.2. *If the cover* $\mathcal{V}$ *of* X *is the union of a family* $\{\mathcal{V}_\lambda : \lambda \in \Lambda\}$ *of enveloping subcollections, and if the cover* $\mathcal{V}_\lambda$ *of* $\cup\mathcal{V}_\lambda$ *can be shrunk for every* $\lambda \in \Lambda$, *then* $\mathcal{V}$ *can be shrunk.*

Proof. For each λ, let $\{A_{V,\lambda} : V \in \mathcal{V}_\lambda\}$ be a shrinking of the cover $\mathcal{V}_\lambda$ of $\cup\mathcal{V}_\lambda$. Note that, since $\mathcal{V}_\lambda$ is enveloping, the sets $A_{V,\lambda}$ are closed not only in $\cup\mathcal{V}_\lambda$ but also in X.

Well-order the index set Λ. For each $V \in \mathcal{V}$, let $\lambda(V) = \inf\{\lambda \in \Lambda : V \in \mathcal{V}_\lambda\}$, and define $A_V = A_{V,\lambda(V)}$. To show that $\{A_V : V \in \mathcal{V}\}$ shrinks $\mathcal{V}$, it will suffice to check that it covers X. So suppose $x \in X$, and let $\gamma = \inf\{\lambda \in \Lambda : x \in \cup\mathcal{V}_\lambda\}$. Then $x \in A_{V,\gamma} \subset V$ for some $V \in \mathcal{V}_\gamma$, and clearly $\lambda(V) = \gamma$ for this V. Hence $x \in A_{V,\lambda(V)} = A_V$, which completes the proof.

3. Proof of Theorem 1.1

In view of Lemmas 2.1 and 2.2, it will suffice to establish Theorem 1.1 for *countable* covers $\mathcal{V}$. This is easily done if each $V \in \mathcal{V}$ actually has a Lindelöf *closure*, for then X is itself Lindelöf and thus paracompact, and *every* open cover of a papacompact space can be shrunk [1, Lemma 5.1.6]. Since Theorem 1.1 only assumes that each $V \in \mathcal{V}$ has a Lindelöf *boundary*, however, the proof is a bit harder, and will be given in several steps. We denote the boundary of a set S by ∂S.

The following lemma slightly generalizes the result that every regular Lindelöf space is normal, and has essentially the same proof.

Lemma 3.1. *If* X *is regular, then any two disjoint, closed subsets* A *and* B *of* X *with Lindelöf boundaries can be separated by disjoint open sets* U *and* V.[2]

Proof. Cover ∂A by open subsets U_n ($n \in N$) of X whose closures miss B, and cover ∂B by open sets V_n in X whose closures miss A. Let $U_n^* = U_n \setminus \cup_{i=1}^n \overline{V}_i$ and let $V_n^* = V_n \setminus \cup_{i=1}^n \overline{U}_i$. Let $U = A^\circ \cup \cup_{n=1}^\infty U_n^*$ and let $V = B^\circ \cup \cup_{n=1}^\infty V_n^*$. These sets have the required properties.

Lemma 3.2. *Theorem* 1.1 *is true if there is a* $V^* \in \mathcal{V}$ *which is dense in* X.

Proof. Let $E = X \setminus V^*$ and let $\mathcal{V}_0 = \mathcal{V} \setminus \{V^*\}$. Then $\{V \cap E: V \in \mathcal{V}_0\}$ is an open cover of the regular Lindelöf (hence paracompact) space E, and thus has a shrinking $\{B_V: V \in \mathcal{V}_0\}$ (see [1, Lemma 5.1.6]). For each $V \in \mathcal{V}_0$, the closed subsets B_V and $X \setminus V$ of X are disjoint and have Lindelöf boundaries, so by Lemma 3.1 there is an open U_V in X such that $B_V \subset U_V$ and $\overline{U}_V \subset V$. Now let

$$A_V = X \setminus \cup\{U_V: V \in \mathcal{V}_0\} \qquad \text{if } V = V^*,$$
$$A_V = \overline{U}_V \qquad \text{if } V \in \mathcal{V}_0.$$

Then $\{A_V: V \in \mathcal{V}\}$ is the required shrinking of $\mathcal{V}$.

Lemma 3.3. *Theorem* 1.1 *is true if* $\mathcal{V}$ *is countable.*

Proof. Write $\mathcal{V} = (V_n)$. Fix $m \in N$. Then $\{V_n \cap \overline{V}_m: n \in N\}$ is an open cover of $\overline{V}_m$, and $\partial_{\overline{V}_m}(V_n \cap \overline{V}_m)$ is a closed subset of the Lindelöf space ∂V_n for all n, so we can apply

[2] By complementation, this is equivalent to the assertion that Theorem 1.1 is true if $\mathcal{V}$ has two elements.

Lemma 3.2 (with X replaced by $\overline{V}_m$ and with V^* replaced by V_m) to obtain a closed cover $\{A_{m,n}: n \in N\}$ of $\overline{V}_m$ such that $A_{mn} \subset V_n$ for all n.

For each n, define $A_n = \cup_{m \le n} A_{mn}$. To show that (A_n) shrinks (V_n), it will suffice to check that it covers X. Suppose $x \in X$, and let $m = \min\{n \in N: x \in V_n\}$. Then $x \in V_m \subset \overline{V}_m$, so $x \in A_{mn} \subset V_n$ for some n. But then $m \le n$ for this n by the definition of m, so $x \in A_n$.

As observed at the beginning of this section, Theorem 1.1 follows immediately from Lemmas 2.1, 2.2 and 3.3.

4. An Application of Theorem 1.1

In this section, we use Theorem 1.1 to prove a result which will be applied in [2], and which answers a question implicitly asked by E. van Douwen in his review of [3] (see Math. Reviews 81 m(1981), no. 54036). It should be remarked that the proof of Corollary 4.1 uses only the special case of Theorem 1.1 where each $V \in \mathcal{V}$ has a Lindelöf closure (rather than just a Lindelöf boundary). Under this stronger hypothesis, the proof of Theorem 1.1 can be significantly shortened; see the first paragraph of Section 3.

Corollary 4.1. *Let* X *be a meta-Lindelöf (resp. σ-metacompact), locally Lindelöf*[3] *regular space, and let* $\mathcal{B}$ *be a base for* X. *Then* X *has a cover* $\mathcal{B}' \subset \mathcal{B}$ *such that the indexed family* $\{\overline{B}: B \in \mathcal{B}'\}$ *is point-countable (resp. σ-point-finite).*

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[3]A space X is *meta-Lindelöf* (resp. *σ-metacompact*) if every open cover of X has a point-countable (resp. σ-point-finite) open refinement, and X is *locally Lindelöf* if every $x \in X$ has a neighborhood V whose closure is Lindelöf.

Proof. Let $\mathcal{U}$ be a cover of X by open subsets with Lindelöf closures, and let $\mathcal{V}$ be a point-countable (resp. σ-point-finite) open refinement of $\mathcal{U}$. Then $\overline{V}$ is Lindelöf for every $V \in \mathcal{V}$, so Theorem 1.1 implies that $\mathcal{V}$ can be shrunk to closed cover $\{A_V: V \in \mathcal{V}\}$. If $V \in \mathcal{V}$, then A_V is Lindelöf, so there is a countable $\mathcal{B}_V \subset \mathcal{B}$ covering A_V such that $\overline{B} \subset V$ for every $B \in \mathcal{B}_V$. The collection $\mathcal{B}' = \cup\{\mathcal{B}_V: V \in \mathcal{V}\}$ is now easily seen to have the required properties.

5. Examples

In this section we give two examples which indicate barriers to strengthening Theorem 1.1. Both examples are based on the modified (and simplified) Tychonoff plank, which is defined as follows.

Let X_1 be an uncountable discrete space. Let X_2 be a countably infinite discrete space, and let $X_1^* = X_1 \cup \{x_1^*\}$ and $X_2^* = X_2 \cup \{x_2^*\}$ be their one-point compactifications. Let $X = (X_1^* \times X_2^*)\setminus\{(x_1^*,x_2^*)\}$, the modified Tychonoff plank. Let $A = \{x_1^*\} \times X_2$ and let $B = X_1 \times \{x_2^*\}$. Then A and B are disjoint, closed, discrete subsets of X which cannot be separated by open sets; more precisely, if $W \supset A$ is open in X, then $(B\setminus\overline{W} \cap B)$ is countable.

Our first example demonstrates the importance of the assumption in Theorem 1.1 that each $V \in \mathcal{V}$ has a Lindelöf boundary.

Example 5.1. A regular, locally compact space X, and a cover $\{V_1,V_2\}$ of X by open sets with discrete boundaries which cannot be shrunk.

Proof. Let X, $A \subset X$ and $B \subset X$ be as above, and let $V_1 = X \backslash A$ and $V_2 = X \backslash B$. Then $\{V_1, V_2\}$ satisfies our requirements; in particular, $\{V_1, V_2\}$ cannot be shrunk because A and B cannot be separated by open sets.

Our second example shows that the cover $\mathcal{V}$ in Theorem 1.1 need not, in general, have the shrinking property considered in Footnote 1. We will say that a cover $\mathcal{V}$ with this property can be *strictly shrunk*.

Example 5.2. A regular, locally compact space X, and a cover $\{V_1, V_2\}$ of X by open subsets with countable, discrete boundaries which cannot be strictly shrunk.

Proof. Again, let X, $A \subset X$ and $B \subset X$ be as above. Let A_1, A_2 be disjoint, infinite subsets of A which cover A, and let $V_i = X \backslash A_i$ $(i = 1,2)$. Then $\{V_1, V_2\}$ satisfies our requirements; in particular, $\{V_1, V_2\}$ cannot be strictly shrunk because neighborhoods W_1 and W_2 of A_1 and A_2 cannot have disjoint closures (since $\overline{W}_1 \cap B$ and $\overline{W}_2 \cap B$ both have countable complements in the uncountable set B).

6. A Related Question

Example 5.1 shows that "Lindelöf" cannot be weakened to "paracompact" in the hypothesis of Theorem 1.1. The following question, however, appears to be open.

Question 6.1. Can every cover of a regular space by open subsets with paracompact (or even metrizable) closures be shrunk?

While we cannot answer Question 6.1, we have the following related result.

Proposition 6.2. *Every countable cover* $\mathcal{V}$ *of a topological space* X *by open subsets with normal closures and countably paracompact boundaries can be shrunk.*

Proof. The proof parallels that of Lemma 3.3: First, our result is established in the special case where some $V^* \in \mathcal{V}$ is dense in X by following the proof of Lemma 3.2 with minor modifications. The general case follows from this special case just as Lemma 3.3 follows from Lemma 3.2.

Remark. Proposition 6.2 reduces to a well-known result of C. H. Dowker [1, Theorem 5.2.3, (i) $\rightarrow$ (iii)] if the whole space X is normal and countably paracompact. While our hyptoheses easily imply that X is normal, they do not imply that X is countably paracompact (as is seen by taking X to be a Dowker space and $\mathcal{V} = \{X\}$).

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