TOPOLOGY PROCEEDINGS

Volume 8, 1983

Pages 51–54

http://topology.auburn.edu/tp/

THE APOSYNDETIC DECOMPOSITION OF HOMOGENEOUS CONTINUA

by

F. Burton Jones

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

THE APOSYNDETIC DECOMPOSITION OF HOMOGENEOUS CONTINUA

F. Burton Jones

The aposyndetic decomposition theorem for homogeneous continua was proved many years ago [3] but the argument given then was tedious and convoluted. Now with the Effros theorem [2] a much simpler and more direct proof is possible. Furthermore, recent use of an extended version has added new interest [4,5].

Theorem. Suppose that M *is a decomposable, homogeneous continuum (=compact, connected metric space). Then there exists a nondegenerate continuous collection* $\mathcal{G}$ *of disjoint subcontinua of* M *filling up* M *such that* (a) $M/\mathcal{G}$ *is a homogeneous, aposyndetic continuum,* (b) *for each* $x \in M$, $L_x \in \mathcal{G}$, *and* (c) *if* $G \in \mathcal{G}$ *and* K *is a subcontinuum of* M *which intersects both* G *and* M - G, *then* $K \supset G$. *(Clearly the elements of* $\mathcal{G}$ *are homogeneous and mutually homeomorphic.)*
[L_x = {y: M *is not aposyndetic at* y *with respect to* x}.]

Proof. If $p \in M$, let G be a proper subcontinuum of M containing p which is maximal with respect to having property (c). To see that G exists, let $\mathcal{H}$ be the set of all proper subcontinua of M containing p and having property (c). If $\mathrm{cl}(\cup\mathcal{H})$ is an element of $\mathcal{H}$, then it is obviously maximal with respect to having property (c). Let q be a point of M at which M is aposyndetic with respect to p. Such a point must exist since M is decomposable and homogeneous. Let N_q be a closed connected neighborhood of q missing p. Clearly N_q intersects no element of $\mathcal{H}$. So $H = cl(\cup\mathcal{H})$ does not contain q and if it does not have property (c), some subcontinuum K of M contains a point of H and a point of M - H but no point of any element of $\mathcal{H}$. Using the Effros theorem, there exists a homeomorphism of M onto itself which moves K so little that it still misses p, intersects M - H but at the same time intersects some element of $\mathcal{H}$. So H is maximal and will do for G.

Now let $\mathcal{G}$ be the collection of all such subcontinua G. It is easy to see that the elements of $\mathcal{G}$ are disjoint and $\cup\mathcal{G} = M$.

Suppose that for some point p of M, L_p does not have property (c). Then there exists a continuum K which contains a point x of L_p, a point y of $M - L_p$ but does not contain L_p. If K does not contain p, then there is a closed connected neighborhood T of y which misses p and the union of Effros images of K intersecting T but covering up a small neighborhood of x would produce a closed connected neighborhood of x missing p. So it must always happen that K contains p and, in fact, the element of $\mathcal{G}$ containing p.

Let U_p denote $M - L_p$. If G_p denotes the element of $\mathcal{G}$ containing p, $U_p \cap G_p = \phi$. So $G_p \subset L_p$. By the above, p cuts $L_p - \{p\}$ from U_p. Hence by Bing's "non-domain-cut-point" theorem [1], $L_p - \{p\}$ (and also L_p) has void interior (relM, of course). So any closed connected neighborhood of a point of L_p intersects U_p. Let p_1 be a point of L_p - K and x be a point of L_{p_1}. If N is a closed connected neighborhood of x, N must contain p_1. Since N also intersects U_p, it must contain p. So $L_{p_1} \subset L_p$. None of the points of $K \cap L_p$ belong to L_{p_1} because then K can be fattened up to make a closed connected neighborhood missing p_1. So $L_{p_1} \subsetneq L_p$. In particular, $p \notin L_{p_1}$.

Let h be a homeomorphism of M onto M such that $h(p) = p_1$. Then $h(L_p) = L_{p_1}$, $h(p_1) = p_2$, $L_{p_2} \subsetneq L_{p_1}$ with $p_1 \notin L_{p_2}, \cdots, h(p_n) = p_{n+1} \cdots$ continues ω times. Let $L = \cap_{i=1}^{\infty} L_{p_i}$ and let p_ω be a point of L. Let x be any point of L_{p_ω} and N be a closed connected neighborhood of x. Since $p_\omega \in L_{p_i}$ and N contains a point of U_{p_i} (in fact, a point of U_p), N must contain p_i. Hence $L_{p_\omega} \subsetneq L_{p_i}$. Let h^ω be a homeomorphism of M onto M that takes p to p_ω. This starts the process all over again. So it goes on without ever stopping and uncountably many times is too much. So $L_p = G_p$ and has property (c).

It is easy to see that $\mathcal{G}$ is upper-semicontinuous and that each element of $\mathcal{G}$ is homogeneous. Since $\mathcal{G}$ is continuous at some one of its elements, the homogeneity forces $\mathcal{G}$ to be a continuous decomposition. Furthermore, if G_1 and G_2 are different elements of $\mathcal{G}$, any closed connected neighborhood of a point of G_1 missing a point of G_2 is a closed connected neighborhood of G_1 missing G_2. Hence $M/\mathcal{G}$ is aposyndetic. Clearly it is homogeneous.

Bibliography

1. R. H. Bing, *Some characterizations of arcs and simple closed curves*, Amer. J. Math. 70 (1948), 497-506.
2. E. G. Effros, *Transformation groups and C*-algebras*, Ann. of Math. 81 (1965), 38-55.
3. F. B. Jones, *On a certain type of homogeneous plane continua*, Proc. Amer. Math. Soc. 6 (1955), 735-740.
4. J. T. Rogers, Jr., *Completely regular mappings and homogeneous, aposyndetic continua*, Can. J. Math. 33 (1981), 450-453.
5. ________, *Decompositions of homogeneous continua*, Pacific J. Math. 99 (1982), 137-144.

University of California

Riverside, California 92521