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## PROPER REFINABLE MAPS ON LOCALLY COMPACT PARACOMPACT SPACES

by

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## PROPER REFINABLE MAPS ON LOCALLY COMPACT PARACOMPACT SPACES

Akira Koyama

### 1. Introduction

In [3], Ford and Rogers defined a map  $f: X \rightarrow Y$  from a compactum  $X$  onto a compactum  $Y$  to be *refinable* if for each  $\epsilon > 0$  there is an  $\epsilon$ -map from  $X$  onto  $Y$  whose distance from  $f$  is less than  $\epsilon$ . Refinable maps are interesting and useful in shape theory and generalized ANR theory. For example, Kato [5] introduced the notion of pseudo-isomorphisms in shape theory and showed that refinable maps between compacta are pseudo-isomorphisms, and Patten [10] proved that for a refinable map  $f: X \rightarrow Y$  between compacta, if  $X$  is an approximative polyhedron, then so is  $Y$ .

Recently Watanabe [11] introduced and investigated approximative shape theory. Then he defined refinable maps for arbitrary spaces and the notion of approximative pseudo-isomorphisms, and showed that refinable maps between compact Hausdorff spaces are approximative pseudo-isomorphisms. That is, by [11], Lemma (26.7), he extended the Kato's result to compact Hausdorff spaces.

In this paper we will show that proper refinable maps between locally compact paracompact spaces are approximative pseudo-isomorphisms, and that the assumption of properness is essential. Other properties of proper refinable maps are examined.

Throughout this paper we will assume that all spaces are Hausdorff spaces and all maps are continuous functions. An ANR means an absolute neighborhood retract for metric spaces.

## 2. Approximative Shape

In this section we will introduce a few notions from approximative shape theory [11], which are needed in this paper.

For a space  $X$  a cover of  $X$  means a *normal open cover* of  $X$  and by  $\text{Cov}(X)$  we denote the set of all covers of  $X$ . For  $\mathcal{U}, \mathcal{U}' \in \text{Cov}(X)$ ,  $\mathcal{U} \geq \mathcal{U}'$  means that  $\mathcal{U}'$  is a refinement of  $\mathcal{U}$ . We say that maps  $f, g: X \rightarrow Y$  are  $\mathcal{V}$ -near for  $\mathcal{V} \in \text{Cov}(Y)$ , written  $(f, g) \leq \mathcal{V}$ , provided every  $x \in X$  admits  $V \in \mathcal{V}$  such that  $f(x), g(x) \in V$ . A system  $(\underline{X}, \underline{\mathcal{U}}) = \{(X_a, \mathcal{U}_a), p_{aa}, A\}$  consisting of pairs of spaces  $X_a$  and their covers  $\mathcal{U}_a$ , and maps  $p_{aa}: X_{a'} \rightarrow X_a$  is called an *approximative inverse system* if it satisfies the following three conditions:

(AI1)  $\underline{X} = \{X_a, p_{aa}, A\}$  is an inverse system over a closure-finite directed set  $A$  in the category  $\text{TOP}$  of spaces and maps,

(AI2)  $p_{aa'}^{-1}(\mathcal{U}_a) \geq \mathcal{U}_{a'}$ , for every  $a' \geq a$  in  $A$ , and

(AI3) for every  $a \in A$  and every  $\mathcal{U} \in \text{Cov}(X_a)$  there is  $a' \geq a$  in  $A$  such that  $p_{aa'}^{-1}(\mathcal{U}) \geq \mathcal{U}_{a'}$ .

Let  $(\underline{Y}, \underline{\mathcal{V}}) = \{(Y_b, \mathcal{V}_b), q_{bb'}, B\}$  be an approximative inverse system. A pair  $\underline{f} = (f, \{f_b\}_{b \in B})$  consisting of a function  $f: B \rightarrow A$  and a collection  $\{f_b\}_{b \in B}$  of maps  $f_b: X_{f(b)} \rightarrow Y_b$  is called an *approximative system map*, written  $\underline{f}: (\underline{X}, \underline{\mathcal{U}}) \rightarrow (\underline{Y}, \underline{\mathcal{V}})$ , provided  $\underline{f}$  satisfies the following two conditions:

(AM1)  $f_b^{-1}(V_b) \supseteq U_{f(b)}$  for every  $b \in B$ , and

(AM2) for every  $b \leq b'$  in  $B$  there is a  $\geq f(b)$ ,  $f(b')$

in  $A$  such that  $(f_b p_{f(b)a}, q_{bb'}, f_{b'} p_{f(b')a}) \leq V_b$ .

Let  $X$  be a space. Let  $(X, \mathcal{U})$  be an approximative inverse system and let  $\underline{p} = \{p_a: X \rightarrow X_a \mid a \in A\}$  be a collection of maps.

We call  $\underline{p}$  an *approximative resolution* of  $X$  if

$p_a = p_{aa}, p_a$ , for every  $a \leq a'$  in  $A$ , and the following conditions are satisfied:

(AR1) for every  $\mathcal{U} \in \text{Cov}(X)$  there is a  $a \in A$  such that  $p_a^{-1}(\mathcal{U}_a) \leq \mathcal{U}$ , and

(AR2) for every  $a \in A$  there is  $a' \geq a$  in  $A$  such that  $p_{aa'}(X_{a'}) \subset \text{st}(p_a(X), \mathcal{U}_a)$ .

Then we will simply denote the approximative resolution by

$\underline{p}: X \rightarrow (X, \mathcal{U})$ . If all  $X_a$  are ANR's (polyhedra), then we call  $\underline{p}$  an *ANR-approximative resolution (polyhedral approximative resolution)*.

Let  $f: X \rightarrow Y$  be a map. Let  $\underline{p}: X \rightarrow (X, \mathcal{U})$  and  $\underline{q}: Y \rightarrow (Y, \mathcal{V})$  be approximative resolutions of  $X$  and  $Y$ , respectively.

Then an approximative system map  $\underline{f}: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$  is an *approximative resolution of  $f$  with respect to  $\underline{p}$  and  $\underline{q}$*  provided that

$(q_b f, f_b p_{f(b)}) \leq V_b$  for every  $b \in B$ .

We call an approximative system map  $\underline{f}: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$  an *approximative pseudo-isomorphism* if for every pair

$(a, b) \in A \times B$  with  $a \geq f(b)$  in  $A$ , there exist  $g(a, b) \geq b$  in  $B$  and a map  $g_{(a,b)}: Y_{g(a,b)} \rightarrow X_a$  such that

$(f_b p_{f(b)a} g_{(a,b)}, q_{bg(a,b)}) \leq \text{st}(V_b)$ , and for every  $b' \geq g(a, b)$  in  $B$  there are  $h(b') \geq a$ ,  $f(b')$  in  $A$  and a map

$h_{b'}: X_{h(b')} \rightarrow Y_b$ , such that  $(g_{(a,b)} q_{g(a,b)b'} h_{b'}, p_{ah(b')}) \leq U_a$ .

Moreover if we can take  $f_{b, p_f(b')h(b')}$  as  $h_{b'}$ , we call  $\underline{f}$  an *approximative isomorphism*.

A map  $f: X \rightarrow Y$  is said to be an *approximative pseudo-isomorphism (approximative isomorphism)* if there exists an approximative pseudo-isomorphism (approximative isomorphism)  $\underline{f}: (\underline{X}, \underline{U}) \rightarrow (\underline{Y}, \underline{V})$  which is an approximative resolution of  $f$  with respect to some ANR-approximative resolutions  $\underline{p}: X \rightarrow (\underline{X}, \underline{U})$   $\underline{q}: Y \rightarrow (\underline{Y}, \underline{V})$  of  $X$  and  $Y$ , respectively.

A space  $X$  is an *approximative polyhedron (AP)* if for every  $\underline{U} \in \text{Cov}(X)$  there exist a polyhedron  $P$  and maps  $f: X \rightarrow P$ ,  $g: P \rightarrow X$  such that  $(gf, l_X) \leq \underline{U}$  (see [9] and [10]).

An approximative inverse system  $(\underline{X}, \underline{U})$  is *approximatively calm* if there exists  $a_0 \in A$  such that for every  $a \geq a_0$  in  $A$  there is  $a' \geq a$  in  $A$  such that, for two maps  $f, g: Z \rightarrow X_a$ , if  $(p_{a_0 a'} f, p_{a_0 a'} g) \leq \underline{U}_{a_0}$ , then  $(p_{aa'} f, p_{aa'} g) \leq \underline{U}_a$ . A space  $X$  is *approximatively calm* if  $X$  admits an ANR-approximative resolution  $\underline{p}: X \rightarrow (\underline{X}, \underline{U})$  such that  $(\underline{X}, \underline{U})$  is approximatively calm (c.f. [2] and [6]).

### 3. Results

A map  $f: X \rightarrow Y$  is a  $\underline{U}$ -map for  $\underline{U} \in \text{Cov}(X)$  provided for every  $y \in Y$ , there is  $U \in \underline{U}$  such that  $f^{-1}(y) \subset U$ . A map  $r: X \rightarrow Y$  is *refinable* if for every  $\underline{U} \in \text{Cov}(X)$  and every  $\underline{V} \in \text{Cov}(Y)$ , there exists a surjective  $\underline{U}$ -map  $f: X \rightarrow Y$  such that  $(r, f) \leq \underline{V}$ . Our definition of refinable maps coincides with the original one by Ford and Rogers [3] in the case of compacta. We remark that refinable maps between locally compact paracompact spaces are proper if and only if they are  $c$ -refinable maps defined in [7].

In the latter part of this paper we assume that *all spaces considered are locally compact paracompact spaces, unless the contrary is specifically indicated.*

*Theorem 1. Every proper refinable map  $f: X \rightarrow Y$  is an approximative pseudo-isomorphism.*

For the proof of Theorem 1 we need some lemmas which are already known in the cases of compacta and compact Hausdorff spaces (see [8] and [11]).

*Lemma 1. Let  $K$  be a locally finite simplicial complex. Then for every closed subset  $A$  of the underlying polyhedron  $|K|$ , there are a subcomplex  $K_0$  of  $K$  and a map  $f: A \rightarrow |K_0|$  such that  $f(A) = |K_0|$  and  $f(A \cap |s|) \subset |s|$  for every  $s \in K$ .*

*Proof.* Put a collection  $S = \{s \in K \mid A \cap |s| \neq \emptyset\}$  of simplexes of  $K$ . Then we define  $S_0 = S \cap K^{(0)}$  and a map  $f_0 = 1_{|S \cap K^{(0)}|}$ .

For an integer  $n \geq 0$  we assume that we already have a subcomplex  $S_n \subset S \cap K^{(n)}$  of  $K$  and a map  $f_n: |S \cap K^{(n)}| \cap A \rightarrow |S_n|$  such that  $S_{n-1} \subset S_n$ ,  $f_n \upharpoonright |S \cap K^{(n-1)}| = f_{n-1}$ ,  $f_n(|S \cap K^{(n)}|) = |S_n|$ ,  $f_n \upharpoonright |s| = 1_{|s|}$  for every  $s \in S \cap K^{(n)}$  with  $A \cap |s| = |s|$ , and  $f_n(A \cap |s|) \subset |s|$  for every  $s \in S \cap K^{(n)}$ .

We decompose the family  $S^{(n+1)} = \{s \in S \cap K^{(n+1)} \mid \dim s = n+1\}$  into the following three subfamilies:

$$S_1^{(n+1)} = \{s \in S^{(n+1)} \mid A \cap |s| = |s|\},$$

$$S_2^{(n+1)} = \{s \in S^{(n+1)} \mid A \cap |s| \neq |s|, s' \in S_n$$

for some face  $s'$  of  $s\}$ , and

$$S_3^{(n+1)} = \{s \in S^{(n+1)} \mid A \cap |s| \neq |s|, s' \notin S_n \text{ for any face } s' \text{ of } s\}.$$

Then we have that  $S^{(n+1)} = S_1^{(n+1)} \cup S_2^{(n+1)} \cup S_3^{(n+1)}$  and  $S_i^{(n+1)} \cap S_j^{(n+1)} = \emptyset$  if  $i \neq j$ . For each  $s \in S_2^{(n+1)}$  we can have a map  $f_s: |s| \cap A \rightarrow |S_n|$  such that  $f_s(|s| \cap A) = \cup\{|s'| \mid s' \in S_n \cap \partial s\}$  and  $f_s \mid |\partial s| \cap A = f_n \mid |\partial s| \cap A$ . For each  $s \in S_3^{(n+1)}$  take a map  $g_s: |s| \cap A \rightarrow |s|$  such that  $g_s(|s| \cap A) = |s'|$  for some face  $s'$  of  $s$ . Then we define  $S_{n+1} = S_n \cup S_1^{(n+1)} \cup \{s' \mid g_s(|s|) = |s'|, s \in S_3^{(n+1)}\}$ , and the map  $f_{n+1}: |S \cap K^{(n+1)}| \cap A \rightarrow |S_{n+1}|$  as follows:

$$\begin{aligned} f_{n+1} \mid |S \cap K^{(n)}| \cap A &= f_n, \\ f_{n+1} \mid |s| &= 1_{|s|} \text{ for each } s \in S_1^{(n+1)}. \\ f_{n+1} \mid |s| \cap A &= f_s \text{ for each } s \in S_2^{(n+1)}, \text{ and} \\ f_{n+1} \mid |s| \cap A &= g_s \text{ for each } s \in S_3^{(n+1)}. \end{aligned}$$

It is clear that  $S_{n+1}$  and  $f_{n+1}$  satisfy the conditions of the induction which replace  $n-1$  and  $n$  with  $n$  and  $n+1$ .

Finally we define the subcomplex  $K_0 = \cup_{n \geq 0} S_n$  and the function  $f: A \rightarrow |K_0|$  by  $f(x) = f_n(x)$  if  $x \in |S \cap K^{(n)}|$  for some  $n \geq 0$ . Since  $K$  is locally finite,  $f$  is well-defined and continuous. Moreover we can easily show that  $K_0$  and  $f$  satisfy the desired conditions.

*Lemma 2.* Let  $f: X \rightarrow P$  be a map from a space  $X$  to a polyhedron  $P$ . Then for each open cover  $\mathcal{U}$  of  $P$  there is an open cover  $\mathcal{V}$  of  $X$  such that for every closed  $\mathcal{V}$ -map  $g: X \rightarrow Y$  from  $X$  onto a space  $Y$ , there exists a map  $h: Y \rightarrow P$  such that  $(hg, f) \leq \mathcal{U}$ .

*Proof.* Take a triangulation  $K$  of  $P$  such that  $\{\text{St}(v:K) \mid v \in K^{(0)}\} \leq \mathcal{U}$ , where  $\text{St}(v:K)$  is the open star

of  $v$  in  $K$ . Then we define  $\mathcal{V} = \{f^{-1}(\text{St}(v:K)) \mid v \in K^{(0)}\}$ .

In order to show that  $\mathcal{V}$  satisfies the required conditions, we take any closed  $\mathcal{V}$ -map  $g: X \rightarrow Y$  from  $X$  onto a space  $Y$ . Since  $Y$  is strongly paracompact, there is a star-finite cover  $\mathcal{W}$  of  $Y$  such that  $g^{-1}(\text{st}(\mathcal{W})) \subseteq \mathcal{V}$  and  $\text{cl}(W)$  is compact for each  $W \in \mathcal{W}$ . Then there are a locally finite simplicial complex  $L$  and a canonical map  $h: Y \rightarrow |L|$  of  $\mathcal{W}$ . Then  $h$  is proper, because  $L$  is locally finite and  $\text{cl}(W)$  is compact for every  $W \in \mathcal{W}$ . Hence  $h(Y)$  is closed in  $|L|$ . By Lemma 1 there are a subcomplex  $L_0$  of  $L$  and a map  $k: h(Y) \rightarrow |L_0|$  such that  $k(h(Y)) = |L_0|$  and  $k(h(Y) \cap |s|) \subset |s|$  for every  $s \in L$ . Then  $k^{-1}(\text{St}(t:L_0)) \subset \text{St}(t:L)$  for every vertex  $t \in L_0$ . Since  $g^{-1}(\text{st}(\mathcal{W})) \subseteq \mathcal{V}$ , for each vertex  $t$  of  $L_0$ , there is a vertex  $\phi(t)$  of  $K$  such that

$$(hg)^{-1}(k^{-1}(\text{St}(t:L_0))) \subset f^{-1}(\text{St}(\phi(t):K)).$$

Then the correspondence  $\phi: L_0^{(0)} \rightarrow K^{(0)}$  induces the simplicial map  $\phi: (|L_0|, L_0) \rightarrow (P, K)$ . It suffices to show that  $(f, (\phi k)g) \subseteq \mathcal{U}$ . For any point  $x \in X$ ,  $khg(x) \in \text{St}(t:L_0)$  for some vertex  $t$  of  $L_0$ . Then by the definition of  $\phi$ ,  $x \in f^{-1}(\text{St}(\phi(t):K))$ . That is,  $f(x) \in \text{St}(\phi(t):K)$ . On the other hand,  $\phi(khg(x)) \in \text{St}(\phi(t):K)$ , because  $\phi$  is simplicial. Hence  $(f, (\phi k)g) \subseteq \mathcal{U}$ .

Since locally compact paracompact spaces are strongly paracompact, by the proof of [9], Theorem 11 and [11], Lemma (3.6), we have the following.

*Lemma 3. Every space  $X$  admits a polyhedral approximative resolution  $\underline{p} = \{p_a \mid a \in A\}: X \rightarrow (X, \mathcal{U}) =$*



$\{(X_a, \mathcal{U}_a), p_{aa}, A\}$  such that for every  $a \in A$ ,  $X_a$  is locally compact and  $p_a$  is a closed map.

*Lemma 4* (Watanabe [11], Theorem (3.2)). Let  $(X, \mathcal{U}) = \{(X_a, \mathcal{U}_a), p_{aa}, A\}$  be an approximative inverse system. A collection  $\underline{p} = \{p_a: X \rightarrow X_a \mid a \in A\}$  of maps induces the approximative resolution  $\underline{p}: X \rightarrow (X, \mathcal{U})$  of  $X$  if and only if it induces the resolution  $\underline{p}: X \rightarrow \underline{X} = \{X_a, p_{aa}, A\}$  of  $X$  in the sense of Mardešić [9]. That is,  $\underline{p}$  satisfies the following conditions:

(R1) Let  $P$  be an AP,  $\mathcal{V}$  a cover of  $P$ . Then for every map  $f: X \rightarrow P$  there are  $a \in A$  and a map  $f_a: X_a \rightarrow P$  such that  $(f_a p_a, f) \leq \mathcal{V}$ , and

(R2) Let  $P$  be an AP,  $\mathcal{V}$  a cover of  $P$ . Then there is a cover  $\mathcal{V}'$  of  $P$  with the following property:

if  $a \in A$  and  $f, f': X_a \rightarrow P$  are maps such that  $(f p_a, f' p_a) \leq \mathcal{V}'$ , there exists  $a' \geq a$  in  $A$  such that  $(f p_{aa'}, f' p_{aa'}) \leq \mathcal{V}$ .

*Proof of Theorem 1.* By Lemma 3 there are polyhedral approximative resolutions  $\underline{p} = \{p_a \mid a \in A\}: X \rightarrow (X, \mathcal{U}) = \{(X_a, \mathcal{U}_a), p_{aa}, A\}$  and  $\underline{q} = \{q_b \mid b \in B\}: Y \rightarrow (Y, \mathcal{V}) = \{(Y_b, \mathcal{V}_b), q_{bb}, B\}$  of  $X$  and  $Y$ , respectively such that for every  $a \in A$  and  $b \in B$ , both  $X_a$  and  $Y_b$  are locally compact and both  $p_a$  and  $q_b$  are closed maps. Let  $\underline{f} = (f, \{f_b\}_{b \in B}): (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$  be an approximative resolution of  $f$  with respect to  $\underline{p}$  and  $\underline{q}$ . Fix any pair  $(a, b) \in A \times B$  with  $a \geq f(b)$  in  $A$ . By Lemma 4 and (AI3) for  $(Y, \mathcal{V})$  there are  $\mathcal{V}_b^1 \in \text{Cov}(Y_b)$  and  $b_1 \geq b$  in  $B$  such that

- (1)  $V_b^1$  satisfies (R2) for  $q$  and  $V_b$ , and
- (2)  $q_{bb_1}^{-1}(V_b^1) \geq \text{st}(V_{b_1})$ .

By (AM2) for  $\underline{f}$ , (AI3) for  $(X, \mathcal{U})$  and Lemma 4 there are  $a_1 \geq a$ ,  $f(b_1)$  in  $A$ , and  $\mathcal{U}_{a_1}^1, \mathcal{U}_{a_1}^2 \in \text{Cov}(X_{a_1})$  such that

- (3)  $(f_b p_{f(b)} a_1, q_{bb_1} f_{b_1} p_{f(b_1)} a_1) \leq V_b$ ,
- (4)  $\text{st}(\mathcal{U}_{a_1}^2) \leq \mathcal{U}_{a_1}^1 \leq \mathcal{U}_{a_1}$ , and
- (5)  $\mathcal{U}_{a_1}^1$  satisfies (R2) for  $p$  and  $\mathcal{U}_{a_1}$ .

Now for  $\mathcal{U}_{a_1}^2 \in \text{Cov}(X_{a_1})$  and the closed map  $p_{a_1} : X \rightarrow X_{a_1}$  we take  $\mathcal{U} \in \text{Cov}(X)$  satisfying the condition of Lemma 2. Put  $V = q_{b_1}^{-1}(V_{b_1}) \in \text{Cov}(Y)$ .

Then there is a surjective  $\mathcal{U}$ -map  $g : X \rightarrow Y$  with  $(f, g) \leq V$ . Since  $X$  is locally compact and  $f$  is proper, we may assume that  $g$  is proper, in particular, closed. Hence by the choosing of  $\mathcal{U} \in \text{Cov}(X)$ , there is a map  $h : Y \rightarrow X_{a_1}$  such that

$$(6) (hg, p_{a_1}) \leq \mathcal{U}_{a_1}^2.$$

Moreover there are  $b_2 \geq b_1$  in  $B$  and a map  $k : Y_{b_2} \rightarrow X_{a_1}$  such that

$$(7) (kq_{b_2}, h) \leq \mathcal{U}_{a_1}^2.$$

Then by (4), (6) and (7) we have that

$$(8) (kq_{b_2} g, p_{a_1}) \leq \mathcal{U}_{a_1}^1.$$

Hence

$$(9) (f_{b_1} p_{f(b_1)} a_1 kq_{b_2} g, f_{b_1} p_{f(b_1)} a_1) \leq V_{b_1}.$$

Since  $\underline{f}$  is an approximative resolution of  $f$  and  $(f, g) \leq V$ , by (9),

$$(10) (f_{b_1} p_{f(b_1)} a_1 kq_{b_2} g, q_{b_1} g) \leq \text{st}(V_{b_1}).$$

Hence by (2) and surjectivity of  $g$

$$(11) \quad (q_{bb_1} f_{b_1} p_{f(b_1)} a_1 k_{q_{b_2}} , q_b) \leq \mathcal{V}_b^1.$$

By (1) and (11) we have  $b_3 \geq b_2$  in  $B$  such that

$$(12) \quad (q_{bb_1} f_{b_1} p_{f(b_1)} a_1 k_{q_{b_2 b_3}} , q_{bb_3}) \leq \mathcal{V}_b.$$

Therefore by (3)

$$(13) \quad (f_b p_{f(b)} a_1 k_{q_{b_2 b_3}} , q_{bb_3}) \leq st(\mathcal{V}_b).$$

Next we take any  $b_4 \geq b_3$  in  $B$ . Then there are  $a_2 \geq a_1$  in  $A$  and a map  $m: X_{a_2} \rightarrow Y_{b_4}$  such that

$$(14) \quad (mp_{a_2} , q_{b_4} g) \leq (k_{q_{b_2 b_4}})^{-1} (\mathcal{U}_{a_1}^2).$$

Then by (7), (6) and (4) we have that

$$(15) \quad (k_{q_{b_2 b_4}} mp_{a_2} , p_{a_1}) \leq \mathcal{U}_{a_1}^1.$$

Hence by (5) there exists  $a_3 \geq a_2$  in  $A$  such that

$$(16) \quad (k_{q_{b_2 b_4}} mp_{a_2 a_3} , p_{a_1 a_3}) \leq \mathcal{U}_{a_1}.$$

Therefore  $\underline{f}$  is an approximative pseudo-isomorphism. This completes the proof of Theorem 1.

By [11], Lemma (26.7) we have the following generalization of the Kato's result to locally compact paracompact spaces.

*Corollary 1. Every proper refinable map  $f: X \rightarrow Y$  is a pseudo-isomorphism.*

An inverse system  $\underline{X} = \{X_a, p_{aa}, A\}$  in HTOP is said to be *calm* if there exists  $a_0 \in A$  satisfying the following condition: every  $a \geq a_0$  in  $A$  there is  $a' \geq a$  in  $A$  such that for any two maps  $f, g: Z \rightarrow X_a$ , from an ANR  $Z$ , if  $p_{a_0 a} f$  is homotopic to  $p_{a_0 a} g$ , then  $p_{aa} f$  is homotopic to  $p_{aa} g$ . A

space  $X$  is *calm* if there is an inverse system  $\underline{X}$  associated with  $X$  such that  $X$  is calm (see [2] and [6]).

By [6], Lemma (2.1), and Corollary 1 we have the following.

*Corollary 2. Let  $f: X \rightarrow Y$  be a proper refinable map. If  $Y$  is calm, then  $f$  induces the shape equivalence.*

We refer to [1] for other results about shape dominations and calmness.

By the same way as the proof of [6], Lemma (2.1), we can easily have the fact: *if an approximative system map  $\underline{f}: (\underline{X}, \underline{U}) \rightarrow (\underline{Y}, \underline{V})$  is an approximative pseudo-isomorphism and  $(\underline{Y}, \underline{V})$  is approximative calm, then  $\underline{f}$  is the approximative isomorphism. Therefore by Theorem 1 we have the next result.*

*Theorem 2. Let  $f: X \rightarrow Y$  be a proper refinable map. If  $Y$  is approximative calm, then  $f$  is the approximative isomorphism.*

By [11], Corollaries (10.8) and (26.15) we have the generalization of Patten [10], Theorem 2.

*Corollary 3. Every proper refinable image of an AP is an AP.*

Next we will show the analogues of Lemma 1 to ANR's.

*Lemma 5. Let  $f: X \rightarrow M$  be a map from a space  $X$  to a locally compact ANR  $M$ . For every  $\underline{U} \in \text{Cov}(M)$  there is  $\underline{V} \in \text{Cov}(X)$  having the following property: for every closed*

$V$ -map  $g: X \rightarrow Y$  from  $X$  onto a space  $Y$ , there is a map  $h: Y \rightarrow M$  such that  $(hg, f) \leq \text{st}(\mathcal{U})$ .

*Proof.* Since  $M$  is a locally compact ANR, there are locally compact polyhedron  $P$ , a proper map  $\phi: M \rightarrow P$  and a map  $\psi: P \rightarrow M$  such that  $(\psi\phi, 1_M) \leq \mathcal{U}$ . For each  $y \in M$  there is an open neighborhood  $O_y$  of  $\phi(y)$  in  $P$  such that  $\{y\} \cup \psi(O_y) \subset U_y$  for some  $U_y \in \mathcal{U}$ . Since  $\phi(M)$  is closed in  $P$ , there is a triangulation  $K$  of  $P$  such that

$$\begin{aligned} \text{St}(K)_{\phi(M)} &= \{\text{St}(v:K) \mid \text{St}(v:K) \cap \phi(M) \neq \emptyset\} \\ &\leq \mathcal{O} = \{O_y \mid y \in M\}. \end{aligned}$$

We define  $V = (\phi f)^{-1}(\text{St}(K)_{\phi(M)})$ .

Now we will show that  $V$  satisfies the required condition. Take any surjective closed  $V$ -map  $g: X \rightarrow Y$ . There is a cover  $\mathcal{W}$  of  $Y$  such that  $g^{-1}(\mathcal{W}) \leq V$ . Then by the same way as the proof of Lemma 3 we have a locally finite simplicial complex  $L$  and a surjective map  $\chi: Y \rightarrow |L|$  such that  $\chi^{-1}(\text{St}(L)) \leq \mathcal{W}$ . Then for each  $t \in L^{(0)}$  there is  $\sigma(t) \in K^{(0)}$  such that  $\text{St}(\sigma(t):K) \cap \phi(M) \neq \emptyset$  and  $(\chi g)^{-1}(\text{St}(t:L)) \subset (\phi f)^{-1}(\text{St}(\sigma(t):K))$ . The correspondence  $\sigma$  induces the simplicial map  $\sigma: |L| \rightarrow |K| = P$ . For each  $x \in X$   $(\chi g)(x) \in \text{St}(t:L)$  for some  $t \in L^{(0)}$ . By definitions there is  $y \in M$  such that

$$(\phi f)(x) \in \text{St}(\sigma(t):K) \subset O_y.$$

Since  $\psi(\text{St}(\sigma(t):K)) \subset \psi(O_y) \subset U_y$  and  $\sigma$  is simplicial,  $\psi\sigma\chi g(x) \in U_y$ . On the other hand,  $f(x) \in U_{f(x)}$  and  $(\phi f)(x) \in O_y \cap O_{f(x)} \subset U_y \cap U_{f(x)} \neq \emptyset$ . Hence  $\psi\sigma\chi g(x), f(x) \in \text{st}(U_{f(x)}: \mathcal{U})$ .

Therefore  $((\psi\sigma\chi)g, f) \leq \text{st}(\mathcal{U})$ .

*Theorem 3. Let  $f: X \rightarrow Y$  be a proper refinable map.*

*If  $X$  is an AP, for every  $V \in \text{Cov}(Y)$  there is a proper map  $g: Y \rightarrow X$  such that  $(fg, l_Y) \leq V$ .*

*Proof.* For every  $V \in \text{Cov}(Y)$  there is  $U \in \text{Cov}(X)$   $\text{st}(U) \leq f^{-1}(V)$  and such that if maps  $\phi, \psi: Z \rightarrow X$  are  $\text{st}(U)$ -near and  $\phi$  is proper,  $\psi$  is also proper. Since  $X$  is an AP and locally compact paracompact, there are a locally compact ANR  $M$ , a proper map  $p: X \rightarrow M$  and a map  $q: M \rightarrow X$  such that  $(qp, l_X) \leq U$ . For each  $x \in X$  there are  $U_x \in U$  and an open neighborhood  $W_x$  of  $f(x)$  in  $M$  such that  $x, qp(x) \in U_x$  and  $g(W_x) \subset U_x$ . Define an open cover  $\mathcal{W} = \{M - f(X)\} \cup \{W_x \mid x \in X\}$  of  $M$ . Let  $\mathcal{U}_0$  be an open cover of  $X$  satisfying the condition of Lemma 1 for  $\mathcal{W} \in \text{Cov}(M)$ . Since  $f$  is refinable, there is a surjective  $\mathcal{U}_0$ -map  $h: X \rightarrow Y$  with  $(f, h) \leq V$ . Then there exists a map  $k: Y \rightarrow M$  such that  $(kh, p) \leq \mathcal{W}$ . By an easy calculation  $(qkh, l_X) \leq \text{st}(U)$ . Since  $\text{st}(U) \leq f^{-1}(V)$  and  $(f, h) \leq V$ ,  $(fqkh, h) \leq \text{st}(V)$ . Hence  $(f(qk), l_Y) \leq \text{st}(V)$ . Notice that  $qkh$  is proper, because  $(qkh, l_X) \leq \text{st}(U)$  and  $l_X$  is proper. Hence  $qk$  is proper.

*Corollary 4. Under the assumption of Theorem 3, if  $Y$  is an ANR, then  $f$  is the proper homotopy domination.*

*Moreover if  $X$  is an ANR,  $f$  is the proper homotopy equivalence.*

This easily follows from the slight modification of the proof of Theorem 3.

We will notice that all above results do not hold unless suitable refinable maps are proper by simple examples.

*Example.* Let  $Y$  be a compactum which is not an AP. Let  $X$  be the space which is equal to  $Y$  as the set, and which has the discrete topology. Then the identity map  $f: X \rightarrow Y$  is refinable. But  $f$  is not an approximative pseudo-isomorphism, because  $X$  is the 0-dimensional locally compact polyhedron.

Similarly if the above  $Y$  is an FANR with the infinite cardinality,  $\text{Sh}(X) \not\cong \text{Sh}(Y)$ . Therefore  $f$  is not even a shape equivalence. That is, Corollary 2, Theorem 2 and Corollary 4 is not valid unless  $f$  is proper.

Finally we point out the relation between refinable maps and cell-like maps.

*Theorem 4.* Let  $f: X \rightarrow Y$  be a proper refinable map between locally compact metric spaces. If  $Y$  is  $\text{LC}^n$  ( $n = 0, 1, 2, \dots$ ), then  $f^{-1}(y)$  is  $\text{AC}^n$  for every  $y \in Y$ .

*In particular, if  $Y$  is an ANR,  $f$  is the cell-like map.*

The proof is a slight modification of [4], Theorem. Hence the detail will be omitted here.

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