TOPOLOGY PROCEEDINGS
Volume 8, 1983
Pages 55–69

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN:	0146-4124

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1. Introduction

In [3], Ford and Rogers defined a map $f: X \to Y$ from a compactum X onto a compactum Y to be *refinable* if for each $\varepsilon > 0$ there is an ε-map from X onto Y whose distance from f is less than ε. Refinable maps are interesting and useful in shape theory and generalized ANR theory. For example, Kato [5] introduced the notion of pseudo-isomorphisms in shape theory and showed that refinable maps between compacta are pseudo-isomorphisms, and Patten [10] proved that for a refinable map $f: X \to Y$ between compacta, if X is an approximative polyhedron, then so is Y.

Recently Watanabe [11] introduced and investigated approximative shape theory. Then he defined refinable maps for arbitrary spaces and the notion of approximative pseudo-isomorphisms, and showed that refinable maps between compact Huasdorff spaces are approximative pseudo-isomorphisms. That is, by [11], Lemma (26.7), he extended the Kato's result to compact Hausdorff spaces.

In this paper we will show that proper refinable maps between locally compact paracompact spaces are approximative pseudo-isomorphisms, and that the assumption of properness is essential. Other properties of proper refinable maps are examined.

Throughout this paper we will assume that all spaces are Hausdorff spaces and all maps are continuous functions. An ANR means an absolute neighborhood retract for metric spaces.

2. Approximative Shape

In this section we will introduce a few notions from approximative shape theory [11], which are needed in this paper.

For a space X a cover of X means a *normal open cover* of X and by Cov(X) we denote the set of all covers of X. For $\mathcal{U}$, $\mathcal{U}' \in$ Cov(X), $\mathcal{U} \geq \mathcal{U}'$ means that $\mathcal{U}'$ is a refinement of $\mathcal{U}$. We say that maps f, g: X → Y are *$\mathcal{V}$-near* for $\mathcal{V} \in$ Cov(Y), written $(f,g) \leq \mathcal{V}$, provided every $x \in X$ admits $V \in \mathcal{V}$ such that $f(x), g(x) \in V$. A system $(\underline{X,\mathcal{U}}) = \{(X_a,\mathcal{U}_a),p_{aa'},A\}$ consisting of pairs of spaces X_a and their covers $\mathcal{U}_a$, and maps $p_{aa'}: X_{a'} \to X_a$ is called an *approximative inverse system* if it satisfies the following three conditions:

(AI1) $\underline{X} = \{X_a,p_{aa'},A\}$ is an inverse system over a closure-finite directed set A in the category TOP of spaces and maps,

(AI2) $p_{aa'}^{-1}(\mathcal{U}_a) \geq \mathcal{U}_{a'}$ for every $a' \geq a$ in A, and

(AI3) for every $a \in A$ and every $\mathcal{U} \in$ Cov(X_a) there is $a' \geq a$ in A such that $p_{aa'}^{-1}(\mathcal{U}) \geq U_{a'}$.

Let $(\underline{Y,\mathcal{V}}) = \{(Y_b,\mathcal{V}_b),q_{bb'},B\}$ be an approximative inverse system. A pair $\underline{f} = (f,\{f_b\}_{b\in B})$ consisting of a function $f: B \to A$ and a collection $\{f_b\}_{b\in B}$ of maps $f_b: X_{f(b)} \to Y_b$ is called an *approximative system map*, written $\underline{f}: (\underline{X,\mathcal{U}}) \to (\underline{Y,\mathcal{V}})$, provided $\underline{f}$ satisfies the following two conditions:

(AM1) $f_b^{-1}(\mathcal{V}_b) \geq \mathcal{U}_{f(b)}$ for every $b \in B$, and

(AM2) for every $b \leq b'$ in B there is $a \geq f(b), f(b')$ in A such that $(f_b p_{f(b)a}, q_{bb'} f_{b'} p_{f(b')a}) \leq \mathcal{V}_b$.

Let X be a space. Let $(\underline{X}, \underline{\mathcal{U}})$ be an approximative inverse system and let $\underline{p} = \{p_a: X \to X_a | a \in A\}$ be a collection of maps. We call $\underline{p}$ an *approximative resolution of* X if $p_a = p_{aa'} p_{a'}$ for every $a \leq a'$ in A, and the following conditions are satisfied:

(AR1) for every $\mathcal{U} \in \mathrm{Cov}(X)$ there is $a \in A$ such that $p_a^{-1}(\mathcal{U}_a) \leq \mathcal{U}$, and

(AR2) for every $a \in A$ there is $a' \geq a$ in A such that $p_{aa'}(X_{a'}) \subset \mathrm{st}(p_a(X), \mathcal{U}_a)$.

Then we will simply denote the approximative resolution by $\underline{p}: X \to (\underline{X}, \underline{\mathcal{U}})$. If all X_a are ANR's (polyhedra), then we call $\underline{p}$ an ANR-*approximative resolution (polyhedral approximative resolution)*.

Let $f: X \to Y$ be a map. Let $\underline{p}: X \to (\underline{X}, \underline{\mathcal{U}})$ and $\underline{q}: Y \to (\underline{Y}, \underline{\mathcal{V}})$ be approximative resolutions of X and Y, respectively. Then an approximative system map $\underline{f}: (\underline{X}, \underline{\mathcal{U}}) \to (\underline{Y}, \underline{\mathcal{V}})$ is an *approximative resolution of* f *with respect to* $\underline{p}$ *and* $\underline{q}$ provided that $(q_b f, f_b p_{f(b)}) \leq \mathcal{V}_b$ for every $b \in B$.

We call an approximative system map $\underline{f}: (\underline{X}, \underline{\mathcal{U}}) \to (\underline{Y}, \underline{\mathcal{V}})$ an *approximative pseudo-isomorphism* if for every pair $(a,b) \in A \times B$ with $a \geq f(b)$ in A, there exist $g(a,b) \geq b$ in B and a map $g_{(a,b)}: Y_{g(a,b)} \to X_a$ such that $(f_b p_{f(b)a} g_{(a,b)}, q_{bg(a,b)}) \leq \mathrm{st}(\mathcal{V}_b)$, and for every $b' \geq g(a,b)$ in B there are $h(b') \geq a, f(b')$ in A and a map $h_{b'}: X_{h(b')} \to Y_{b'}$ such that $(g_{(a,b)} q_{g(a,b)b'} h_{b'}, p_{ah(b')}) \leq \mathcal{U}_a$.

Moreover if we can take $f_{b'}p_{f(b')h(b')}$ as $h_{b'}$, we call $\underline{f}$ an *approximative isomorphism.*

A map f: X → Y is said to be an *approximative pseudo-isomorphism (approximative isomorphism)* if there exists an approximative pseudo-isomorphism (approximative isomorphism) $\underline{f}$: $(\underline{X}, \mathcal{U}) \to (\underline{Y}, \mathcal{V})$ which is an approximative resolution of f with respect to some ANR-approximative resolutions $\underline{p}$: $X \to (\underline{X}, \mathcal{U})$ $\underline{q}$: $Y \to (\underline{Y}, \mathcal{V})$ of X and Y, respectively.

A space X is an *approximative polyhedron* (AP) if for every $\mathcal{U} \in \mathrm{Cov}(X)$ there exist a polyhedron P and maps f: X → P, g: P → X such that $(gf, 1_X) \leq \mathcal{U}$ (see [9] and [10]).

An approximative inverse system $(\underline{X}, \mathcal{U})$ is *approximatively calm* if there exists $a_0 \in A$ such that for every $a \geq a_0$ in A there is $a' \geq a$ in A such that, for two maps f,g: $Z \to X_{a'}$, if $(p_{a_0a'}f, p_{a_0a'}g) \leq \mathcal{U}_{a_0}$, then $(p_{aa'}f, p_{aa'}g) \leq \mathcal{U}_a$. A space X is *approximatively calm* if X admits an ANR-approximative resolution $\underline{p}$: $X \to (\underline{X}, \mathcal{U})$ such that $(\underline{X}, \mathcal{U})$ is approximatively calm (c.f. [2] and [6]).

3. Results

A map f: X → Y is a *$\mathcal{U}$-map* for $\mathcal{U} \in \mathrm{Cov}(X)$ provided for every $y \in Y$, there is $U \in \mathcal{U}$ such that $f^{-1}(y) \subset U$. A map r: X → Y is *refinable* if for every $\mathcal{U} \in \mathrm{Cov}(X)$ and every $\mathcal{V} \in \mathrm{Cov}(Y)$, there exists a surjective $\mathcal{U}$-map f: X → Y such that $(r, f) \leq \mathcal{V}$. Our definition of refinable maps coincides with the original one by Ford and Rogers [3] in the case of compacta. We remark that refinable maps between locally compact paracompact spaces are proper if and only if they are c-refinable maps defined in [7].

In the latter part of this paper we assume that *all spaces considered are locally compact paracompact spaces*, unless the contrary is specifically indicated.

Theorem 1. *Every proper refinable map* $f: X \to Y$ *is an approximative pseudo-isomorphism.*

For the proof of Theorem 1 we need some lemmas which are already known in the cases of compacta and compact Hausdorff spaces (see [8] and [11]).

Lemma 1. *Let* K *be a locally finite simplicial complex. Then for every closed subset* A *of the underlying polyhedron* $|K|$, *there are a subcomplex* K_0 *of* K *and a map* $f: A \to |K_0|$ *such that* $f(A) = |K_0|$ *and* $f(A \cap |s|) \subset |s|$ *for every* $s \in K$.

Proof. Put a collection $S = \{s \in K | A \cap |s| \neq \emptyset\}$ of simplexes of K. Then we define $S_0 = S \cap K^{(0)}$ and a map $f_0 = 1_{|S \cap K^{(0)}|}$.

For an integer $n \geq 0$ we assume that we already have a subcomplex $S_n \subset S \cap K^{(n)}$ of K and a map $f_n: |S \cap K^{(n)}| \cap A \to |S_n|$ such that $S_{n-1} \subset S_n$, $f_n \mid |S \cap K^{(n-1)}| = f_{n-1}$, $f_n(|S \cap K^{(n)}|) = |S_n|$, $f_n \mid |s| = 1_{|s|}$ for every $s \in S \cap K^{(n)}$ with $A \cap |s| = |s|$, and $f_n(A \cap |s|) \subset |s|$ for every $s \in S \cap K^{(n)}$.

We decompose the family $S^{(n+1)} = \{s \in S \cap K^{(n+1)} \mid \dim s = n+1\}$ into the following three subfamilies:

$$S_1^{(n+1)} = \{s \in S^{(n+1)} \mid A \cap |s| = |s|\},$$

$$S_2^{(n+1)} = \{s \in S^{(n+1)} \mid A \cap |s| \neq |s|, s' \in S_n \text{ for some face } s' \text{ of } s\}, \text{ and}$$

$$S_3^{(n+1)} = \{s \in S^{(n+1)} \mid A \cap |s| \neq |s|,\ s' \not\in S_n \text{ for any face } s' \text{ of } s\}.$$

Then we have that $S^{(n+1)} = S_1^{(n+1)} \cup S_2^{(n+1)} \cup S_3^{(n+1)}$ and $S_i^{(n+1)} \cap S_j^{(n+1)} = \emptyset$ if $i \neq j$. For each $s \in S_2^{(n+1)}$ we can have a map $f_s\colon |s| \cap A \to |S_n|$ such that $f_s(|s| \cap A) = \cup\{|s'| \mid s' \in S_n \cap \partial s\}$ and $f_s \mid |\partial s| \cap A = f_n \mid |\partial s| \cap A$. For each $s \in S_3^{(n+1)}$ take a map $g_s\colon |s| \cap A \to |s|$ such that $g_s(|s| \cap A) = |s'|$ for some face s' of s. Then we define $S_{n+1} = S_n \cup S_1^{(n+1)} \cup \{s' \mid g_s(|s|) = |s'|,\ s \in S_3^{(n+1)}\}$, and the map $f_{n+1}\colon |S \cap K^{(n+1)}| \cap A \to |S_{n+1}|$ as follows:

$$f_{n+1} \mid |S \cap K^{(n)}| \cap A = f_n,$$
$$f_{n+1} \mid |s| = 1_{|s|} \text{ for each } s \in S_1^{(n+1)}.$$
$$f_{n+1} \mid |s| \cap A = f_s \text{ for each } s \in S_2^{(n+1)}, \text{ and}$$
$$f_{n+1} \mid |s| \cap A = g_s \text{ for each } s \in S_3^{(n+1)}.$$

It is clear that S_{n+1} and f_{n+1} satisfy the conditions of the induction which replace $n-1$ and n with n and $n+1$.

Finally we define the subcomplex $K_0 = \cup_{n \geq 0} S_n$ and the function $f\colon A \to |K_0|$ by $f(x) = f_n(x)$ if $x \in |S \cap K^{(n)}|$ for some $n \geq 0$. Since K is locally finite, f is well-defined and continuous. Moreover we can easily show that K_0 and f satisfy the desired conditions.

Lemma 2. *Let* $f\colon X \to P$ *be a map from a space* X *to a polyhedron* P. *Then for each open cover* $\mathcal{U}$ *of* P *there is an open cover* $\mathcal{V}$ *of* X *such that for every closed* $\mathcal{V}$*-map* $g\colon X \to Y$ *from* X *onto a space* Y, *there exists a map* $h\colon Y \to P$ *such that* $(hg, f) \leq \mathcal{U}$.

Proof. Take a triangulation K of P such that $\{St(v{:}K) \mid v \in K^{(0)}\} \leq \mathcal{U}$, where $St(v{:}K)$ is the open star of v in K. Then we define $\mathcal{V} = \{f^{-1}(St(v:K)) \mid v \in K^{(0)}\}$.

In order to show that $\mathcal{V}$ satisfies the required conditions, we take any closed $\mathcal{V}$-map $g: X \to Y$ from X onto a space Y. Since Y is strongly paracompact, there is a star-finite cover $\mathcal{W}$ of Y such that $g^{-1}(st(\mathcal{W})) \leq \mathcal{V}$ and cl(W) is compact for each $W \in \mathcal{W}$. Then there are a locally finite simplicial complex L and a canonical map $h: Y \to |L|$ of $\mathcal{W}$. Then h is proper, because L is locally finite and cl(W) is compact for every $W \in \mathcal{W}$. Hence h(Y) is closed in $|L|$. By Lemma 1 there are a subcomplex L_0 of L and a map $k: h(Y) \to |L_0|$ such that $k(h(Y)) = |L_0|$ and $k(h(Y) \cap |s|) \subset |s|$ for every $s \in L$. Then $k^{-1}(St(t:L_0)) \subset St(t:L)$ for every vertex $t \in L_0$. Since $g^{-1}(st(\mathcal{W})) \leq \mathcal{V}$, for each vertex t of L_0, there is a vertex $\phi(t)$ of K such that

$$(hg)^{-1}(k^{-1}(St(t:L_0))) \subset f^{-1}(St(\phi(t):K)).$$

Then the correspondence $\phi: L_0^{(0)} \to K^{(0)}$ induces the simplicial map $\phi: (|L_0|, L_0) \to (P,K)$. It suffices to show that $(f,(\phi kh)g) \leq \mathcal{U}$. For any point $x \in X$, $khg(x) \in St(t:L_0)$ for some vertex t of L_0. Then by the definition of ϕ, $x \in f^{-1}(St(\phi(t):K)$. That is, $f(x) \in St(\phi(t):K)$. On the other hand, $\phi(khg(x)) \in St(\phi(t):K)$, because ϕ is simplicial. Hence $(f,(\phi kh)g) \leq \mathcal{U}$.

Since locally compact paracompact spaces are strongly paracompact, by the proof of [9], Theorem 11 and [11], Lemma (3.6), we have the following.

Lemma 3. *Every space* X *admits a polyhedral approximative resolution* $\underline{p} = \{p_a \mid a \in A\}: X \to (\underline{X}, \underline{\mathcal{U}}) =$

$\{(X_a, \mathcal{U}_a), p_{aa'}, A\}$ *such that for every* $a \in A$, X_a *is locally compact and* p_a *is a closed map.*

Lemma 4 (Watanabe [11], Theorem (3.2)). *Let* $(\underline{X}, \underline{\mathcal{U}}) = \{(X_a, \mathcal{U}_a), p_{aa'}, A\}$ *be an approximative inverse system. A collection* $\underline{p} = \{p_a: X \to X_a \mid a \in A\}$ *of maps induces the approximative resolution* $\underline{p}: X \to (\underline{X}, \underline{\mathcal{U}})$ *of* X *if and only if it induces the resolution* $\underline{p}: X \to \underline{X} = \{X_a, p_{aa'}, A\}$ *of* X *in the sense of Mardešić* [9]. *That is,* p *satisfies the following conditions:*

(R1) *Let* P *be an* AP, $\mathcal{V}$ *a cover of* P. *Then for every map* $f: X \to P$ *there are* $a \in A$ *and a map* $f_a: X_a \to P$ *such that* $(f_a p_a, f) \leq \mathcal{V}$, *and*

(R2) *Let* P *be an* AP, $\mathcal{V}$ *a cover of* P. *Then there is a cover* $\mathcal{V}'$ *of* P *with the following property:*

if $a \in A$ *and* $f, f': X_a \to P$ *are maps such that* $(fp_a, f'p_a) \leq \mathcal{V}'$, *there exists* $a' \geq a$ *in* A *such that* $(fp_{aa'}, f'p_{aa'}) \leq \mathcal{V}$.

Proof of Theorem 1. By Lemma 3 there are polyhedral approximative resolutions $\underline{p} = \{p_a \mid a \in A\}: X \to (\underline{X}, \underline{\mathcal{U}}) = \{(X_a, \mathcal{U}_a), p_{aa'}, A\}$ and $\underline{q} = \{q_b \mid b \in B\}: Y \to (\underline{Y}, \underline{\mathcal{V}}) = \{(Y_b, \mathcal{V}_b), q_{bb'}, B\}$ of X and Y, respectively such that for every $a \in A$ and $b \in B$, both X_a and Y_b are locally compact and both p_a and q_b are closed maps. Let $\underline{f} = (f, \{f_b\}_{b \in B})$: $(\underline{X}, \underline{\mathcal{U}}) \to (\underline{Y}, \underline{\mathcal{V}})$ be an approximative resolution of f with respect to $\underline{p}$ and $\underline{q}$. Fix any pair (a,b) A × B with $a \geq f(b)$ in A. By Lemma 4 and (AI3) for $(\underline{Y}, \underline{\mathcal{V}})$ there are $\mathcal{V}_b^1 \in \mathrm{Cov}(Y_b)$ and $b_1 \geq b$ in B such that

(1) $\mathcal{V}_b^1$ satisfies (R2) for $\underline{q}$ and $\mathcal{V}_b$, and

(2) $q_{bb_1}^{-1}(\mathcal{V}_b^1) \geq st(\mathcal{V}_{b_1})$.

By (AM2) for $\underline{f}$, (AI3) for $(\underline{X},\mathcal{U})$ and Lemma 4 there are $a_1 \geq a$, $f(b_1)$ in A, and $\mathcal{U}_{a_1}^1$, $\mathcal{U}_{a_1}^2 \in Cov(X_{a_1})$ such that

(3) $(f_b p_{f(b)a_1}, q_{bb_1} f_{b_1} p_{f(b_1)a_1}) \leq \mathcal{V}_b$,

(4) $st(\mathcal{U}_{a_1}^2) \leq \mathcal{U}_{a_1}^1 \leq \mathcal{U}_{a_1}$, and

(5) $\mathcal{U}_{a_1}^1$ satisfies (R2) for $\underline{p}$ and $\mathcal{U}_{a_1}$.

Now for $\mathcal{U}_{a_1}^2 \in Cov(X_{a_1})$ and the closed map $p_{a_1}: X \to X_{a_1}$ we take $\mathcal{U} \in Cov(X)$ satisfying the condition of Lemma 2. Put $\mathcal{V} = q_{b_1}^{-1}(\mathcal{V}_{b_1}) \in Cov(Y)$.

Then there is a surjective $\mathcal{U}$-map $g: X \to Y$ with $(f,g) \leq \mathcal{V}$. Since X is locally compact and f is proper, we may assume that g is proper, in particular, closed. Hence by the chosing of $\mathcal{U} \in Cov(X)$, there is a map $h: Y \to X_{a_1}$ such that

(6) $(hg, p_{a_1}) \leq \mathcal{U}_{a_1}^2$.

Moreover there are $b_2 \geq b_1$ in B and a map $k: Y_{b_2} \to X_{a_1}$ such that

(7) $(kq_{b_2}, h) \leq \mathcal{U}_{a_1}^2$.

Then by (4), (6) and (7) we have that

(8) $(kq_{b_2}g, p_{a_1}) \leq \mathcal{U}_{a_1}^1$.

Hence

(9) $(f_{b_1}p_{f(b_1)a_1}kq_{b_2}g, f_{b_1}p_{f(b_1)}) \leq \mathcal{V}_{b_1}$.

Since $\underline{f}$ is an approximative resolution of f and $(f,g) \leq \mathcal{V}$, by (9),

(10) $(f_{b_1}p_{f(b_1)a_1}kq_{b_2}g, q_{b_1}g) \leq st(\mathcal{V}_{b_1})$.

Hence by (2) and surjectivity of g

(11) $(q_{bb_1}f_{b_1}p_{f(b_1)a_1}kq_{b_2}, q_b) \leq \mathcal{V}_b^1$.

By (1) and (11) we have $b_3 \geq b_2$ in B such that

(12) $(q_{bb_1}f_{b_1}p_{f(b_1)a_1}kq_{b_2b_3}, q_{bb_3}) \leq \mathcal{V}_b$.

Therefore by (3)

(13) $(f_b p_{f(b)a_1}kq_{b_2b_3}, q_{bb_3}) \leq st(\mathcal{V}_b)$.

Next we take any $b_4 \geq b_3$ in B. Then there are $a_2 \geq a_1$ in A and a map $m: X_{a_2} \to Y_{b_4}$ such that

(14) $(mp_{a_2}, q_{b_4}g) \leq (kq_{b_2b_4})^{-1}(\mathcal{U}_{a_1}^2)$.

Then by (7), (6) and (4) we have that

(15) $(kq_{b_2b_4}mp_{a_2}, p_{a_1}) \leq \mathcal{U}_{a_1}^1$.

Hence by (5) there exists $a_3 \geq a_2$ in A such that

(16) $(kq_{b_2b_4}mp_{a_2a_3}, p_{a_1a_3}) \leq \mathcal{U}_{a_1}$.

Therefore $\underline{f}$ is an approximative pseudo-isomorphism. This completes the proof of Theorem 1.

By [11], Lemma (26.7) we have the following generalization of the Kato's result to locally compact paracompact spaces.

Corollary 1. *Every proper refinable map* f: X → Y *is a pseudo-isomorphism.*

An inverse system $\underline{X} = \{X_a, p_{aa'}, A\}$ in HTOP is said to be *calm* if there exists $a_0 \in A$ satisfying the following condition: every $a \geq a_0$ in A there is $a' \geq a$ in A such that for any two maps $f, g: Z \to X_{a'}$ from an ANR Z, if $p_{a_0a'}f$ is homotopic to $p_{a_0a'}g$, then $p_{aa'}f$ is homotopic to $p_{aa'}g$. A space X is *calm* if there is an inverse system $\underline{X}$ associated with X such that X is calm (see [2] and [6]).

By [6], Lemma (2.1), and Corollary 1 we have the following.

Corollary 2. *Let* f: X → Y *be a proper refinable map. If* Y *is calm, then* f *induces the shape equivalence.*

We refer to [1] for other results about shape dominations and calmness.

By the same way as the proof of [6], Lemma (2.1), we can easily have the fact: *if an approximative system map* $\underline{f}$: $(\underline{X}, \mathcal{U}) \to (\underline{Y}, \mathcal{V})$ *is an approximative pseudo-isomorphism and* $(\underline{Y}, \mathcal{V})$ *is approximative calm, then* $\underline{f}$ *is the approximative isomorphism.* Therefore by Theorem 1 we have the next result.

Theorem 2. *Let* f: X → Y *be a proper refinable map. If* Y *is approximative calm, then* f *is the approximative isomorphism.*

By [11], Corollaries (10.8) and (26.15) we have the generalization of Patten [10], Theorem 2.

Corollary 3. *Every proper refinable image of an* AP *is an* AP.

Next we will show the analogues of Lemma 1 to ANR's.

Lemma 5. *Let* f: X → M *be a map from a space* X *to a locally compact* ANR M. *For every* $\mathcal{U} \in$ Cov(M) *there is* $\mathcal{V} \in$ Cov(X) *having the following property: for every closed*

$\mathcal{V}$*-map* $g: X \to Y$ *from* X *onto a space* Y*, there is a map* $h: Y \to M$ *such that* $(hg, f) \le st(\mathcal{U})$.

Proof. Since M is a locally compact ANR, there are locally compact polyhedron P, a proper map $\phi: M \to P$ and a map $\psi: P \to M$ such that $(\psi\phi, 1_M) \le \mathcal{U}$. For each $y \in M$ there is an open neighborhood O_y of $\phi(y)$ in P such that $\{y\} \cup \psi(O_y) \subset U_y$ for some $U_y \in \mathcal{U}$. Since $\phi(M)$ is closed in P, there is a triangulation K of P such that

$$St(K)_{\phi(M)} = \{St(v:K) \mid St(v:K) \cap \phi(M) \neq \emptyset\}$$
$$\le \mathcal{O} = \{O_y \mid y \in M\}.$$

We define $\mathcal{V} = (\phi f)^{-1}(St(K)_{\phi(M)})$.

Now we will show that $\mathcal{V}$ satisfies the required condition. Take any surjective closed $\mathcal{V}$-map $g: X \to Y$. There is a cover $\mathcal{W}$ of Y such that $g^{-1}(\mathcal{W}) \le \mathcal{V}$. Then by the same way as the proof of Lemma 3 we have a locally finite simplicial complex L and a surjective map $\chi: Y \to |L|$ such that $\chi^{-1}(St(L)) \le \mathcal{W}$. Then for each $t \in L^{(0)}$ there is $\sigma(t) \in K^{(0)}$ such that $St(\sigma(t):K) \cap \phi(M) \neq \emptyset$ and $(\chi g)^{-1}(St(t:L)) \subset (\phi f)^{-1}(St(\sigma(t):K))$. The correspondence σ induces the simplicial map $\sigma: |L| \to |K| = P$. For each $x \in X$ $(\chi g)(x) \in St(t:L)$ for some $t \in L^{(0)}$. By definitions there is $y \in M$ such that

$$(\phi f)(x) \in St(\sigma(t):K) \subset O_y.$$

Since $\psi(St(\sigma(t):K)) \subset \psi(O_y) \subset U_y$ and σ is simplicial, $\psi\sigma\chi g(x) \in U_y$. On the other hand, $f(x) \in U_{f(x)}$ and $(\phi f)(x) \in O_y \cap O_{f(x)} \subset U_y \cap U_{f(x)} \neq \emptyset$. Hence

$$\psi\sigma\chi g(x), f(x) \in st(U_{f(x)}: \mathcal{U}).$$

Therefore $((\psi\sigma\chi)g, f) \le st(\mathcal{U})$.

Theorem 3. *Let* $f: X \to Y$ *be a proper refinable map. If* X *is an* AP, *for every* $\mathcal{V} \in \mathrm{Cov}(Y)$ *there is a proper map* $g: Y \to X$ *such that* $(fg, 1_Y) \le \mathcal{V}$.

Proof. For every $\mathcal{V} \in \mathrm{Cov}(Y)$ there is $\mathcal{U} \in \mathrm{Cov}(X)$ $\mathrm{st}(\mathcal{U}) \le f^{-1}(\mathcal{V})$ and such that if maps $\phi, \psi: Z \to X$ are $\mathrm{st}(\mathcal{U})$-near and ϕ is proper, ψ is also proper. Since X is an AP and locally compact paracompact, there are a locally compact ANR M, a proper map $p: X \to M$ and a map $q: M \to X$ such that $(qp, 1_X) \le \mathcal{U}$. For each $x \in X$ there are $U_x \in \mathcal{U}$ and an open neighborhood W_x of $f(x)$ in M such that $x, qp(x) \in U_x$ and $g(W_x) \subset U_x$. Define an open cover $\mathcal{W} = \{M-f(X)\} \cup \{W_x \mid x \in X\}$ of M. Let $\mathcal{U}_0$ be an open cover of X satisfying the condition of Lemma 1 for $\mathcal{W} \in \mathrm{Cov}(M)$. Since f is refinable, there is a surjective $\mathcal{U}_0$-map $h: X \to Y$ with $(f,h) \le \mathcal{V}$. Then there exists a map $k: Y \to M$ such that $(kh, p) \le \mathcal{W}$. By an easy calculation $(qkh, 1_X) \le \mathrm{st}(\mathcal{U})$. Since $\mathrm{st}(\mathcal{U}) \le f^{-1}(\mathcal{V})$ and $(f,h) \le \mathcal{V}$, $(fqkh, h) \le \mathrm{st}(\mathcal{V})$. Hence $(f(qk), 1_Y) \le \mathrm{st}(\mathcal{V})$. Notice that qkh is proper, because $(qkh, 1_X) \le \mathrm{st}(\mathcal{U})$ and 1_X is proper. Hence qk is proper.

Corollary 4. *Under the assumption of* Theorem 3, *if* Y *is an* ANR, *then* f *is the proper homotopy domination.*

Moreover if X *is an* ANR, f *is the proper homotopy equivalence.*

This easily follows from the slight modification of the proof of Theorem 3.

We will notice that all above results do not hold unless suitable refinable maps are proper by simple examples.

Example. Let Y be a compactum which is not an AP. Let X be the space which is equal to Y as the set, and which has the discrete topology. Then the identity map f: X → Y is refinable. But f is not an approximative pseudo-isomorphism, because X is the 0-dimensional locally compact polyhedron.

Similarily if the above Y is an FANR with the infinite cardinality, Sh(X) $\not\leq$ Sh(Y). Therefore f is not even a shape equivalence. That is, Corollary 2, Theorem 2 and Corollary 4 is not valid unless f is proper.

Finally we point out the relation between refinable maps and cell-like maps.

Theorem 4. *Let* f: X → Y *be a proper refinable map between locally compact metric spaces. If* Y *is* LC^n (n = 0,1,2,····), *then* $f^{-1}(y)$ *is* AC^n *for every* y ∈ Y.

In particular, if Y *is an* ANR, f *is the cell-like map.*

The proof is a slight modification of [4], Theorem. Hence the detail will be omitted here.

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