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HOMOGENEOUS CONTINUA AND CONTINUOUS DECOMPOSITIONS

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The study of homogeneity in one-dimensional continua has been an area of significant interest and activity in recent years, as evidenced in part by the large number of talks about homogeneity presented at this conference. Around 1975 the study of homogeneous continua was enhanced by the discovery of the applicability of a theorem of Effros [11] concerning topological transformation groups. More recent results have made use of a number of techniques, including uniformization of maps on intervals [32], [33], covering spaces and bundles [38], [39], [40], and techniques more commonly found in arguments involving fixed points [15].

In this paper we will review recent work on the relation between continuous decompositions and the construction of homogeneous continua. We will also present discussion and conjecture about a possible classification of one-dimensional homogeneous continua. The details of the arguments for the individual results presented here are being published elsewhere [26], [27], [28].

For convenience we will separately treat those continua which are planar and those which are not.

Homogeneous Plane Continua

The simple closed curve is obviously a homogeneous plane continuum and was long conjectured to be the only such

space until in 1948 Bing [2], [30], showed that the pseudo-arc is also homogeneous. This opened the way for the existence of other homogeneous plane continua. In 1955 Jones [19] introduced continuous decompositions into the subject by showing that every decomposable, homogeneous, plane continuum has a continuous decomposition into mutually homeomorphic, homogeneous, non-separating continua with decomposition space a simple closed curve.

This led Bing and Jones [5] to show that there does exist a circle of pseudo-arcs, i.e. a homogeneous, circle-like, continuum with a continuous decomposition into pseudo-arcs such that the decomposition space is a simple closed curve. They also showed that this continuum is unique. There were now three known non-degenerate, homogeneous, plane continua. No others have been discovered subsequently, and results are tending increasingly to suggest that these are the only such continua. Each of the above examples is circle-like. After Fearnley and Rogers [12], [34], showed that the pseudo-circle, an hereditarily indecomposable, circle-like, non-chainable, plane continuum, is non-homogeneous, Burgess [6] was able to show that the above three continua are the only homogeneous, circle-like, plane continua.

Hagopian [13] and Jones [21] have shown that every indecomposable, homogeneous, plane continuum is hereditarily indecomposable, and Rogers [38] has shown that such a continuum cannot separate the plane. Thus every nondegenerate, homogeneous, plane continuum must be in one of the following five classes:

- 1) simple closed curve,
- 2) pseudo-arc,
- 3) circle of pseudo-arcs,
- 4) tree-like, non-chainable continua, or
- 5) circles of continua of type 4.

As indicated above, there are at present no known homogeneous continua of types 4 or 5. It also remains unknown whether the existence of a homogeneous continuum of type 4 implies the existence of a homogeneous continuum of type 5. Much recent work on classifying homogeneous plane continua has centered on investigating possible continua of type 4. It is known that such homogeneous continuum must be:

- 1) tree-like,
- 2) neither chainable nor almost chainable [4], [23],
- 3) hereditarily indecomposable [13], [21],
- 4) have span zero [31],
- 5) weakly chainable [32],
- 6) contain non-chainable subcontinua [23],
- 7) be "infinite-branched" and "infinite-junctioned" [23].

These properties are not independent in that it is known that some subsets of them imply others. These properties are also very similar to ones which an hereditarily equivalent continuum [9] other than the two known ones (i.e. the arc and the pseudo-arc) would have to satisfy. Thus it is prudent to keep in mind the question by Jones [20] of what is the relationship between homogeneity and hereditary equivalence in continua? No formal relationship is at

present known, but the continua being investigated are in the two cases frequently similar.

If a homogeneous continuum M of type 4 contains a non-degenerate chainable subcontinuum, then that subcontinuum is a pseudo-arc [3]. By homogeneity and hereditary indecomposability, using ϵ -maps as in the Effros theorem [11], one can show that there exist subcontinua of M which are maximal with respect to each of their nondegenerate proper subcontinua being pseudo-arcs. These maximal subcontinua are themselves homogeneous and thus are pseudo-arcs [23]. Since M is homogeneous and hereditarily indecomposable, these maximal subcontinua form a continuous decomposition of M into pseudo-arcs [36]. The decomposition space is also a homogeneous continuum [36] of type 4, and, unless there are non-chainable subcontinua of M which in the decomposition space become chainable, then the decomposition space is a homogeneous continuum with no chainable subcontinua [24], [25]. The author has recently been able to prove the following theorem [26], which shows that the decomposition cannot introduce new chainable continua.

Theorem 1. If X is an hereditarily indecomposable continuum with a continuous decomposition G into chainable subcontinua such that X/G is chainable, then X is chainable (i.e. the pseudo-arc of pseudo-arcs is unique).

Thus if there exists a homogeneous continuum of type 4, there exists one with no non-degenerate chainable subcontinua. What about one with non-degenerate chainable

subcontinua? Theorem 4 (stated later in this paper) shows that one can always mimic the Bing-Jones construction [5] to obtain a homogeneous one-dimensional continuum \tilde{M} with a continuous decomposition G into pseudo-arcs such that \tilde{M}/G is any prechosen one-dimensional homogeneous continuum M . If M is planar, then \tilde{M} can also be chosen to be planar. Thus we have the following theorem.

Theorem 2. If there exists a homogeneous, plane continuum of type 4, then there exist at least two distinct such continua--

one with no nondegenerate, chainable subcontinua (i.e. shrink all pseudo-arcs to points), and one with all small subcontinua being pseudo-arcs (i.e. expand every point to a pseudo-arc).

Thus in studying other possible homogeneous plane continua, it is not sufficient to investigate only local structure or the structure of small subcontinua. One must also look at the global properties of the continuum. There were previously indications of this fact, since there are many continua which are locally homeomorphic to the pseudo-arc but which are not homogeneous, thus reiterating the not always fully appreciated fact that homogeneity is not a local property.

The arguments above on decomposition into pseudo-arcs do not require all of the properties of a continuum of type 4. In particular they do not require that such a continuum be planar. Theorem 2 applies equally well for any continua

which are homogeneous, hereditarily indecomposable, and non-chainable.

The division of homogeneous plane continua into the five classes listed above (possibly some of which are empty) also applies to any one-dimensional homogeneous continuum which can be embedded in a 2-manifold. Hagopian [16] has recently shown that every homogeneous continuum which is properly embeddable in a connected 2-manifold can be embedded either in an annulus or in a Möbius strip. If M can be embedded in a Möbius strip but not in an annulus then M has a continuous decomposition into mutually homeomorphic, homogeneous, tree-like planar continua (i.e. continua of type 4) such that the decomposition space is a simple closed curve. Thus M is a non-planar, homogeneous continuum of type 5.

Non-Planar Homogeneous Continua

For non-planar, homogeneous, one-dimensional continua we will first look at circle-like continua and then at the more general case. The first non-planar, circle-like, homogeneous continua described were the solenoids of van Dantzig [10]. These are obtained as inverse limits of simple closed curves with covering maps as bonding maps. Rogers [35] used the construction of the circle of pseudo-arcs by Bing and Jones [5] to show that for each solenoid one can obtain a homogeneous solenoid of pseudo-arcs. Hagopian and Rogers [17] then extended Burgess' [6] earlier classification to show that every homogeneous, circle-like continuum is either a solenoid, a pseudo-arc, or a solenoid

of pseudo-arcs (considering a simple closed curve as a 1-adic solenoid).

However this result left open a couple of questions before a complete classification of homogeneous circle-like continua was at hand. The solenoids of pseudo-arcs constructed by Rogers [35] were obtained as inverse limits of circles of pseudo-arcs with covering maps as bonding maps, just as solenoids are obtained from circles. The classification by Hagopian and Rogers [17] showed only that a homogeneous, circle-like continuum other than a solenoid or a pseudo-arc admits a continuous decomposition into pseudo-arcs with the decomposition space a solenoid. The continua constructed by Rogers admit such a decomposition, but it was not immediately evident that every continuum with such a decomposition could be obtained from Rogers' construction. There was thus a question of uniqueness.

1) [17] If X_1 and X_2 are two solenoids of pseudo-arcs with decompositions to homeomorphic solenoids, are X_1 and X_2 homeomorphic?

Further, while the continua constructed by Rogers were homogeneous, it was not immediately evident that a circle-like continuum with a continuous decomposition as in the characterization by Hagopian and Rogers was homogeneous.

2) [17] Is every solenoid of pseudo-arcs (in the sense of Hagopian and Rogers [17]) homogeneous?

The author has recently shown [27] that both of these questions have affirmative answers. The proof makes heavy

use of the arguments and constructions of Bing and Jones [5]. It essentially consists of three steps.

Lemma 1 [27]. Every Cantor set of pseudo-arcs is homeomorphic to the product of a Cantor set with a pseudo-arc.

Lemma 2 [27]. Every Cantor set of arcs of pseudo-arcs is homeomorphic to the product of a Cantor set with an arc of pseudo-arcs. Furthermore, homeomorphisms between the two end pairs of Cantor sets of pseudo-arcs of two Cantor sets of arcs of pseudo-arcs can be extended as long as components match up appropriately.

A solenoid can be obtained by identification on the end Cantor sets of two Cantor sets of arcs. The particular solenoid is determined by the identifications. The above two lemmas show that any two solenoids of pseudo-arcs with decompositions to the same solenoid can be mapped homeomorphically onto each other by splitting the solenoid into two Cantor sets of arcs, lifting the ends of these, extending to maps on Cantor sets of pseudo-arcs, and then extending over the intervening arcs of pseudo-arcs.

Theorem 3 [27]. For a given solenoid S there is a unique solenoid of pseudo-arcs P with S as decomposition space.

Thus the classification of homogeneous, circle-like continua is now complete. Still of interest is whether this class includes other interesting classes of homogeneous

continua. For example, every known non-degenerate, homogeneous continuum which is either planar, atriodic, tree-like, hereditarily decomposable, or hereditarily indecomposable is also circle-like. Must this be true for any non-degenerate, homogeneous continuum in one of these five classes?

That there are other one-dimensional, homogeneous continua became apparent when Anderson showed [1] that the Menger universal curve is homogeneous. Case then showed [8] that there exist solenoidal inverse limits of Menger curves which are homogeneous. The following theorem shows that there are still other homogeneous, one-dimensional continua patterned on earlier constructions.

Theorem 4 [28]. If M is a homogeneous, one-dimensional continuum, then there exists a homogeneous, one-dimensional continuum \tilde{M} with a continuous decomposition G into pseudo-arcs such that \tilde{M}/G is homeomorphic to M .

The proof of the above theorem combines techniques of Mioduszewski [29] on mappings of inverse limits and of Bing and Jones [5] on arcs of pseudo-arcs. Crucial in this proof and in the arguments described previously is the fact that any two pseudo-arcs which are close together (in terms of Hausdorff distance) are homeomorphically close together.

In the construction in the above proof, the decomposition elements are terminal continua (i.e. every subcontinuum of M is either \tilde{M} contained in a decomposition element or is a

sum of such elements). This is related to the question of whether \tilde{M} is unique.

Question [28]. If \tilde{M}_1 and \tilde{M}_2 are two (homogeneous) one-dimensional continua with continuous decompositions G_1 and G_2 into pseudo-arcs such that \tilde{M}_1/G_1 and \tilde{M}_2/G_2 are homeomorphic, then must \tilde{M}_1 and \tilde{M}_2 be homeomorphic?

This question has an affirmative answer if all of the decomposition elements in each of G_1 and G_2 are terminal continua. The previous arguments for uniqueness also required additional assumptions. For the uniqueness of the pseudo-arc of pseudo-arcs we assumed hereditary indecomposability. For the uniqueness of the solenoids of pseudo-arcs we assumed the continua were circle-like.

The decomposition constructed in the proof of Theorem 4 has the property that every homeomorphism of M can be lifted to a homeomorphism of \tilde{M} which respects decomposition elements. In the previous discussion we have mentioned decompositions into pseudo-arcs almost exclusively. What about other continua?

Question [28]. For what non-degenerate continua besides the pseudo-arc can one always obtain decompositions as in Theorem 4?

It can be shown that for every one-dimensional continuum X there is a one-dimensional continuum \hat{X} with a continuous decomposition H into Menger universal curves such that \hat{X}/H is homeomorphic to X . In a few cases \hat{X} is even homogeneous

for certain homogeneous X . However \hat{X} does not in general have the property about lifting homeomorphisms of X mentioned earlier. Rogers [37] has used results of Wilson [41] to show that if one has a continuous decomposition of a homogeneous, one-dimensional continuum where all homeomorphisms preserve decomposition elements, then the decomposition elements are acyclic.

Burgess [7] has presented the following conjectured classification of homogeneous, one-dimensional continua.

- 1) solenoids (including simple closed curves)
- 2) psuedo-arc
- 3) Menger universal curve
- 4) Case continua
- 5) solenoids of pseudo-arcs
- 6) Menger curve of pseudo-arcs
- 7) Case continua of pseudo-arcs
- 8) homogeneous, tree-like, non-chainable continua
- 9) replace "pseudo-arcs" in types 5, 6, and 7 with "continua of type 8."

At present, examples of types 1-7 are known. There are no currently known continua of types 8 or 9 and several people have conjectured that none exist. Minc and Rogers have recently shown that there are homogeneous, non-Case, "double solenoid" inverse limits of Menger curves. It has also been conjectured that there is additional pathology possible in homogeneous continua obtained by other solenoidal constructions.

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