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by

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The study of homogeneity in one-dimensional continua
has been an area of significant interest and activity in
recent years, as evidenced in part by the large number of
talks about homogeneity presented at this conference.
Around 1975 the study of homogeneous continua was enhanced
by the discovery of the applicability of a theorem of Effros
[11] concerning topological transformation groups. More
recent results have made use of a number of techniques,
including uniformization of maps on intervals [32], [33],
covering spaces and bundles [38], [39], [40], and techniques
more commonly found in arguments involving fixed points [15].

In this paper we will review recent work on the rela-
tion between continuous decompositions and the construction
of homogeneous continua. We will also present discussion
and conjecture about a possible classification of one-
dimensional homogeneous continua. The details of the argu-
ments for the individual results presented here are being
published elsewhere [26], [27], [28].

For convenience we will separately treat those continua
which are planar and those which are not.

Homogeneous Plane Continua

The simple closed curve is obviously a homogeneous
plane continuum and was long conjectured to be the only such

space until in 1948 Bing [2], [30], showed that the pseudo-
arc is also homogeneous. This opened the way for the exis-
tence of other homogeneous plane continua. In 1955 Jones
[19] introduced continuous decompositions into the subject
by showing that every decomposable, homogeneous, plane con-
tinuum has a continuous decomposition into mutually homeo-
morphic, homogeneous, non-separating continua with decompo-
sition space a simple closed curve.

This led Bing and Jones [5] to show that there does
exist a circle of pseudo-arcs, i.e. a homogeneous, circle-
like, continuum with a continuous decomposition into pseudo-
arcs such that the decomposition space is a simple closed
curve. They also showed that this continuum is unique.
There were now three known non-degenerate, homogeneous,
plane continua. No others have been discovered subsequently,
and results are tending increasingly to suggest that these
are the only such continua. Each of the above examples is
circle-like. After Fearnley and Rogers [12], [34], showed
that the pseudo-circle, an hereditarily indecomposable,
circle-like, non-chainable, plane continuum, is non-homo-
geneous, Burgess [6] was able to show that the above three
continua are the only homogeneous, circle-like, plane con-
tinua.

Hagopian [13] and Jones [21] have shown that every
indecomposable, homogeneous, plane continuum is hereditarily
indecomposable, and Rogers [38] has shown that such a con-
tinuum cannot separate the plane. Thus every nondegenerate,
homogeneous, plane continuum must be in one of the following
five classes:

1) simple closed curve,

2) pseudo-arc,

3) circle of pseudo-arcs,

4) tree-like, non-chainable continua, or

5) circles of continua of type 4.

As indicated above, there are at present no known
homogeneous continua of types 4 or 5. It also remains
unknown whether the existence of a homogeneous continuum
of type 4 implies the existence of a homogeneous continuum
of type 5. Much recent work on classifying homogeneous
plane continua has centered on investigating possible
continua of type 4. It is known that such homogeneous con-
tinuum must be:

1) tree-like,

2) neither chainable nor almost chainable [4], [23],

3) hereditarily indecomposable [13], [21],

4) have span zero [31],

5) weakly chainable [32],

6) contain non-chainable subcontinua [23],

7) be "infinite-branched" and "infinite-junctioned" [23].

These properties are not independent in that it is
known that some subsets of them imply others. These pro-
perties are also very similar to ones which an hereditarily
equivalent continuum [9] other than the two known ones (i.e.
the arc and the pseudo-arc) would have to satisfy. Thus
it is prudent to keep in mind the question by Jones [20]
of what is the relationship between homogeneity and heredi-
tary equivalence in continua? No formal relationship is at

present known, but the continua being investigated are in the
two cases frequently similar.

If a homogeneous continuum M of type 4 contains a
non-degenerate chainable subcontinuum, then that subcon-
tinuum is a pseudo-arc [3]. By homogeneity and hereditary
indecomposability, using ε-maps as in the Effros theorem
[11], one can show that there exist subcontinua of M which
are maximal with respect to each of their nondegenerate
proper subcontinua being pseudo-arcs. These maximal sub-
continua are themselves homogeneous and thus are pseudo-arcs
[23]. Since M is homogeneous and hereditarily indecomposable,
these maximal subcontinua form a continuous decomposition
of M into pseudo-arcs [36]. The decomposition space is
also a homogeneous continuum [36] of type 4, and, unless
there are non-chainable subcontinua of M which in the
decomposition space become chainable, then the decomposition
space is a homogeneous continuum with no chainable subcon-
tinua [24], [25]. The author has recently been able to
prove the following theorem [26], which shows that the
decomposition cannot introduce new chainable continua.

Theorem 1. *If* X *is an hereditarily indecomposable*
continuum with a continuous decomposition G *into chainable*
subcontinua such that X/G *is chainable, then* X *is chainable*
(i.e. the pseudo-arc of pseudo-arcs is unique).

Thus if there exists a homogeneous continuum of type 4,
there exists one with no non-degenerate chainable subcon-
tinua. What about one with non-degenerate chainable

subcontinua?   Theorem 4 (stated later in this paper) shows

that one can always mimic the Bing-Jones construction [5]

to obtain a homogeneous one-dimensional continuum $\tilde{M}$ with a

continuous decomposition G into pseudo-arcs such that $\tilde{M}/G$

is any prechosen one-dimensional homogeneous continuum M.

If M is planar, then $\tilde{M}$ can also be chosen to be planar.

Thus we have the following theorem.

   Theorem 2.   If there exists a homogeneous, plane con-

tinuum of type 4, then there exist at least two distinct

such continua--

   one with no nondegenerate, chainable subcontinua

   (i.e. shrink all pseudo-arcs to points), and

   one with all small subcontinua being pseudo-arcs

   (i.e. expand every point to a pseudo-arc).

   Thus in studying other possible homogeneous plane

continua, it is not sufficient to investigate only local

structure or the structure of small subcontinua.   One must

also look at the global properties of the continuum.

There were previously indications of this fact, since there

are many continua which are locally homeomorphic to the

pseudo-arc but which are not homogeneous, thus reiterating

the not always fully appreciated fact that homogeneity is

not a local property.

   The arguments above on decomposition into pseudo-arcs

do not require all of the properties of a continuum of type

4.   In particular they do not require that such a continuum

be planar.   Theorem 2 applies equally well for any continua

which are homogeneous, hereditarily indecomposable, and non-chainable.

The division of homogeneous plane continua into the five classes listed above (possibly some of which are empty) also applies to any one-dimensional homogeneous continuum which can be embedded in a 2-manifold. Hagopian [16] has recently shown that every homogeneous continuum which is properly embeddable in a connected 2-manifold can be embedded either in an annulus or in a Möbius strip. If M can be embedded in a Möbius strip but not in an annulus then M has a continuous decomposition into mutually homeomorphic, homogeneous, tree-like planar continua (i.e. continua of type 4) such that the decomposition space is a simple closed curve. Thus M is a non-planar, homogeneous continuum of type 5.

Non-Planar Homogeneous Continua

For non-planar, homogeneous, one-dimensional continua we will first look at circle-like continua and then at the more general case. The first non-planar, circle-like, homogeneous continua described were the solenoids of van Dantzig [10]. These are obtained as inverse limits of simple closed curves with covering maps as bonding maps. Rogers [35] used the construction of the circle of pseudo-arcs by Bing and Jones [5] to show that for each solenoid one can obtain a homogeneous solenoid of pseudo-arcs. Hagopian and Rogers [17] then extended Burgess' [6] earlier classification to show that every homogeneous, circle-like continuum is either a solenoid, a pseudo-arc, or a solenoid

of pseudo-arcs (considering a simple closed curve as a
1-adic solenoid).

However this result left open a couple of questions
before a complete classification of homogeneous circle-like
continua was at hand. The solenoids of pseudo-arcs con-
structed by Rogers [35] were obtained as inverse limits of
circles of pseudo-arcs with covering maps as bonding maps,
just as solenoids are obtained from circles. The classi-
fication by Hagopian and Rogers [17] showed only that a
homogeneous, circle-like continuum other than a solenoid
or a pseudo-arc admits a continuous decomposition into
pseudo-arcs with the decomposition space a solenoid. The
continua constructed by Rogers admit such a decomposition,
but it was not immediately evident that every continuum
with such a decomposition could be obtained from Rogers'
construction. There was thus a question of uniqueness.

1) [17] If X_1 and X_2 are two solenoids of pseudo-arcs
with decompositions to homeomorphic solenoids, are X_1 and
X_2 homeomorphic?

Further, while the continua constructed by Rogers were
homogeneous, it was not immediately evident that a circle-
like continuum with a continuous decomposition as in the
characterization by Hagopian and Rogers was homogeneous.

2) [17] Is every solenoid of pseudo-arcs (in the
sense of Hagopian and Rogers [17]) homogeneous?

The author has recently shown [27] that both of these
questions have affirmative answers. The proof makes heavy

use of the arguments and constructions of Bing and Jones [5].
It essentially consists of three steps.

   *Lemma 1 [27]. Every Cantor set of pseudo-arcs is
homeomorphic to the product of a Cantor set with a pseudo-
arc.*

   *Lemma 2 [27]. Every Cantor set of arcs of pseudo-arcs
is homeomorphic to the product of a Cantor set with an arc
of pseudo-arcs. Furthermore, homeomorphisms between the two
end pairs of Cantor sets of pseudo-arcs of two Cantor sets
of arcs of pseudo-arcs can be extended as long as components
match up appropriately.*

   A solenoid can be obtained by identification on the
end Cantor sets of two Cantor sets of arcs. The particular
solenoid is determined by the identifications. The above
two lemmas show that any two solenoids of pseudo-arcs with
decompositions to the same solenoid can be mapped homeo-
morphically onto each other by splitting the solenoid into
two Cantor sets of arcs, lifting the ends of these, extending
to maps on Cantor sets of pseudo-arcs, and then extending
over the intervening arcs of pseudo-arcs.

   *Theorem 3 [27]. For a given solenoid S there is a
unique solenoid of pseudo-arcs P with S as decomposition
space.*

   Thus the classification of homogeneous, circle-like
continua is now complete. Still of interest is whether
this class includes other interesting classes of homogeneous

continua. For example, every known non-degenerate, homo-
geneous continuum which is either planar, atriodic, tree-
like, hereditarily decomposable, or hereditarily indecom-
posable is also circle-like. Must this be true for any
non-degenerate, homogeneous continuum in one of these
five classes?

That there are other one-dimensional, homogeneous
continua became apparent when Anderson showed [1] that the
Menger universal curve is homogeneous. Case then showed
[8] that there exist solenoidal inverse limits of Menger
curves which are homogeneous. The following theorem shows
that there are still other homogeneous, one-dimensional
continua patterned on earlier constructions.

Theorem 4 [28]. *If* M *is a homogeneous, one-dimensional
continuum, then there exists a homogeneous, one-dimensional
continuum* $\tilde{M}$ *with a continuous decomposition* G *into pseudo-
arcs such that* $\tilde{M}/G$ *is homeomorphic to* M.

The proof of the above theorem combines techniques of
Mioduszewski [29] on mappings of inverse limits and of Bing
and Jones [5] on arcs of pseudo-arcs. Crucial in this
proof and in the arguments described previously is the fact
that any two pseudo-arcs which are close together (in
terms of Hausdorff distance) are homeomorphically close
together.

In the construction in the above proof, the decomposi-
tion elements are terminal continua (i.e. every subcontinuum
of M is either contained in a decomposition element or is a

sum of such elements).   This is related to the question of whether $\tilde{M}$ is unique.

Question [28].   If $\tilde{M}_1$ and $\tilde{M}_2$ are two (homogeneous) one-dimensional continua with continuous decompositions G_1 and G_2 into pseudo-arcs such that $\tilde{M}_1/G_1$ and $\tilde{M}_2/G_2$ are homeomorphic, then must $\tilde{M}_1$ and $\tilde{M}_2$ be homeomorphic?

This question has an affirmative answer if all of the decomposition elements in each of G_1 and G_2 are terminal continua.   The previous arguments for uniqueness also required additional assumptions.   For the uniqueness of the pseudo-arc of pseudo-arcs we assumed hereditary indecomposability.   For the uniqueness of the solenoids of pseudo-arcs we assumed the continua were circle-like.

The decomposition constructed in the proof of Theorem 4 has the property that every homeomorphism of M can be lifted to a homeomorphism of $\tilde{M}$ which respects decomposition elements.   In the previous disucssion we have mentioned decompositions into pseudo-arcs almost exclusively.   What about other continua?

Question [28].   For what non-degenerate continua besides the pseudo-arc can one always obtain decompositions as in Theorem 4?

It can be shown that for every one-dimensional continuum X there is a one-dimensional continuum $\hat{X}$ with a continuous decomposition H into Menger universal curves such that $\hat{X}/H$ is homeomorphic to X.   In a few cases $\hat{X}$ is even homogeneous

for certain homogeneous X.   However $\hat{X}$ does not in general
have the property about lifting homeomorphisms of X mentioned
earlier.   Rogers [37] has used results of Wilson [41] to
show that if one has a continuous decomposition of a homo-
geneous, one-dimensional continuum where all homeomorphisms
preserve decomposition elements, then the decomposition
elements are acyclic.

Burgess [7] has presented the following conjectured
classification of homogeneous, one-dimensional continua.

1) solenoids (including simple closed curves)

2) psuedo-arc

3) Menger universal curve

4) Case continua

5) solenoids of pseudo-arcs

6) Menger curve of pseudo-arcs

7) Case continua of pseudo-arcs

8) homogeneous, tree-like, non-chainable continua

9) replace "pseudo-arcs" in types 5, 6, and 7 with
"continua of type 8."

At present, examples of types 1-7 are known.   There
are no currently known continua of types 8 or 9 and several
people have conjectured that none exist.   Minc and Rogers
have recently shown that there are homogeneous, non-Case,
"double solenoid" inverse limits of Menger curves.   It has
also been conjectured that there is additional pathology
possible in homogeneous continua obtained by other solenoidal
constructions.

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