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SUM AND DECOMPOSITION THEOREMS FOR CONTINUA IN CLASS W[1]

James Francis Davis

In this paper we investigate decomposable atriodic continua and their inverse images. We apply the results of this investigation to prove a sum theorem for continua in class W. We also use these results to prove that atriodic continua which admit a monotone upper semicontinuous decomposition to an arc, each element of which is in class W, are in class W.

1. Definitions

All spaces considered in this paper are metric. By a *continuum* we mean a compact connected space. A *mapping* is a continuous function. If f is a mapping defined on a space X and $A \subset X$, denote the restriction of f to A by $f|A$.

Suppose M is a continuum, H is a subcontinuum of M and f is a mapping of a continuum onto M. Following Ingram [5] we say that f *is confluent with respect to* H (respectively, *weakly confluent with respect to* H) provided that for each (resp., some) component, L, of $f^{-1}(H)$, $f(L) = H$. We say that f is *confluent* (resp., *weakly confluent*) provided f is confluent (resp., weakly confluent) with respect to each subcontinuum of M. The continuum M *is in class* W provided

[1]With the exception of Theorem 6, the results in this paper are from the author's doctoral dissertation, written under the direction of W. T. Ingram at the University of Houston.

that for each mapping f of a continuum onto M, f is weakly confluent.

A continuum is a *triod* provided it contains a subcontinuum whose complement has at least three components. A continuum is *atriodic* if it contains no triod. The continuum M is *unicoherent* provided that if A and B are subcontinua of M such that $M = A \cup B$ then $A \cap B$ is connected.

2. Decomposable Atriodic Continua

Throughout sections 2 and 3 we adopt the following *standing hypothesis*: Suppose that H and K are continua which intersect, $X = H \cup K$, X is atriodic, $H \cap K$ is connected, and that each of H and K contains a point which is not in the other.

Proposition 1. *If* T_1 *and* T_2 *are subcontinua of* H *which intersect* $H \cap K$, $T_1 = \overline{T_1 - (T_1 \cap K)}$, *and* $T_2 = \overline{T_2 - (T_2 \cap K)}$, *then either* $T_1 - (T_1 \cap K) \subset T_2 - (T_2 \cap K)$ *or* $T_2 - (T_2 \cap K) \subset T_1 - (T_1 \cap K)$ *and hence either* $T_1 \subset T_2$ *or* $T_2 \subset T_1$.

Proof. Suppose that neither $T_1 - (T_1 \cap K)$ nor $T_2 - (T_2 \cap K)$ is contained in the other. Let $M_1 = T_1 \cup (H \cap K)$ and $M_2 = T_2 \cup (H \cap K)$. Then $M_1 \cup M_2 \cup K$ is the union of three continua which have a point in common (any point of $H \cap K$) and no one of them is contained in the union of the other two. Thus $H \cup K$ contains a triod by Sorgenfrey's Theorem, [9, Theorem 1.8, p. 443]. This is a contradiction. Hence one of $T_1 - (T_2 \cap K)$ and $T_2 - (T_1 \cap K)$ is contained in the other, and therefore either $T_1 \subset T_2$ or $T_2 \subset T_1$.

Proposition 2. *If* T *is a subcontinuum of* H ∪ K *containing a point of* H - (H ∩ K), *then* T - (T ∩ K) *is connected and hence* $\overline{T - (T \cap K)}$ *is a continuum.*

Proof. Suppose that A is a component of T - (T ∩ K). Since A contains no point of K, $A \subset \bar{A} - (\bar{A} \cap K)$. Suppose a is a point in $\bar{A} - (\bar{A} \cap K)$ which is not in A. Then a is a limit point of A and a is not in K. Thus A ∪ {a} is a connected subset of T - (T ∩ K) ([6, Theorem 29, p. 11]) containing A properly. This is inconsistent with the fact that A is a component of T - (T ∩ K). Thus $A = \bar{A} - (\bar{A} - K)$. Suppose that T - (T ∩ K) has two components, A and B. Then, from the above, $\bar{A} = \overline{\bar{A} - (\bar{A} \cap K)}$ and $\bar{B} = \overline{\bar{B} - (\bar{B} \cap K)}$, and it follows from Proposition 1 that one of $A = \bar{A} - (\bar{A} \cap K)$ and $B = \bar{B} - (\bar{B} \cap K)$ is a subset of the other. This is a contradiction since A and B are components of T - (T ∩ K) and do not intersect.

As a special case of Proposition 2 we have the following:

Proposition 3. *Each of* $\overline{H - (H \cap K)}$ *and* $\overline{K - (H \cap K)}$ *is a continuum.*

Proposition 4. *If* T *is a subcontinuum of* X *which intersects both* H - (H ∩ K) *and* K - (H ∩ K), *then* H ∩ K ⊂ T.

Proof. By Proposition 2, each of T - (T ∩ K) and T - (T ∩ H) has just one component. Suppose H ∩ K is not a subset of T. Let x be a point in T - (T ∩ K) and let T_o be the component of T ∩ H containing x. By [6, Theorem 50, p. 18] T_o contains a point, p, which is a limit point of T - (T ∩ H). By Proposition 2, $\overline{T - (T \cap H)}$ is a continuum. Thus T_o, $\overline{T - (T \cap H)}$ and $H \cap K$ are three continua with a point, p, in common such that no one of them is contained in the union of the other two. This implies that X contains a triod by [9, Theorem 1.8, p. 443], a contradiction.

3. Pre-Images of Decomposable Atriodic Continua

Suppose, in addition to the standing hypothesis, that Y is a continuum and that f is a mapping of Y onto X. Let $\Lambda_{-1} = f^{-1}(H - (H \cap K))$, $\Lambda_1 = f^{-1}(K - (H \cap K))$ and $\Lambda_o = f^{-1}(H \cap K)$.

Lemma 1. *Suppose that* $n = -1$ *or* $n = 1$, M *is a subcontinuum of* Y, *and* x *is a point in* Λ_o *which is a limit point of* $M \cap \Lambda_n$. *Then there exist a subcontinuum* L *of* $\overline{M \cap \Lambda_n}$ *which contains* x *and a monotonic sequence* $A_1 \supset A_2 \supset A_3 \supset \cdots$ *of closed subsets of* $\overline{M \cap \Lambda_n}$ *such that*

(1) $\cap_{i>0} A_i = L$,

(2) A_i *intersects* Λ_n *for* $i = 1,2,3,\cdots$,

(3) $f(A_i)$ *is connected for* $i = 1,2,3,\cdots$, *and*

(4) $\overline{A_i \cap \Lambda_n} = A_i$ *for* $i = 1,2,3,\cdots$.

Proof. Let $x_1, x_2, x_3, \cdots$ be a sequence of points in $M \cap \Lambda_n$ which converges to x. For $i = 1,2,\cdots$ let F(i) be the closure of the component, E(i), of $M \cap \Lambda_n$ which contains x_i. Let $F_1(1), F_1(2), F_1(3), \cdots$ be a subsequence of $F(1), F(2), F(3), \cdots$ which has a sequential limiting set, L, which contains x and is a continuum [6, Theorems 58, 59, pp. 23-24]. Let $E_1(1), E_1(2), E_1(3), \cdots$ be the corresponding subsequence of $E(1), E(2), E(3), \cdots$. If k is a positive integer, $E_1(k)$ has a limit point in Λ_o by [6, Theorem 50, p. 18], and thus $F_1(k) = \overline{E_1(k)}$ intersects Λ_o.

Let $A_k = \overline{(\cup_{i \geq k} F_1(i))} = (\cup_{i \geq k} F_1(i)) \cup L$ for $k = 1,2,3,\cdots$. The sequence $A_1, A_2, A_3, \cdots$ is a monotonic sequence of closed subsets of $\overline{M \cap \Lambda_n}$ and $\cap_{k>0} A_k = L$. If k is a positive integer then $F_1(k)$ intersects Λ_n, hence A_k intersects Λ_n.

Suppose that k is a positive integer and that s and t are in $f(\cup_{i \geq k} F_1(i))$. Then there are integers i, j $\geq$ k such that s is in $F_1(i)$ and t is in $F_1(j)$. Now $\overline{F_1(i) \cap \Lambda_n} = \overline{E_1(i)} = F_1(i)$ and $\overline{F_1(j) \cap \Lambda_n} = \overline{E_1(j)} = F_1(j)$. Thus

$$\overline{f(F_1(i)) - (f(F_1(i)) \cap (H \cap K))} = f(F_1(i)),$$

and

$$\overline{f(F_1(j)) - (f(F_1(j)) \cap (H \cap K))} = f(F_1(j)).$$

Now each of $F_1(i)$ and $F_1(j)$ intersects Λ_o so that each of $f(F_1(i))$ and $f(F_1(j))$ intersects $H \cap K$. Hence, by Proposition 1, either $f(F_1(i)) \subset f(F_1(j))$ or $f(F_1(j)) \subset f(F_1(i))$. Each of $f(F_1(i))$ and $f(F_1(j))$ is a subset of $f(\cup_{i \geq k} F_1(i))$ and each is a continuum. Thus in either of the above cases there is a continuum which contains both s and t and which is a subset of $f(\cup_{i \geq k} F_1(i))$. Thus $f(\cup_{i \geq k} F_1(i))$ is connected. Hence $f(A_k) = f(\overline{\cup_{i \geq k} F_1(i)}) = \overline{f(\cup_{i \geq k} F_1(i))}$ is connected.

Suppose that k is a positive integer and r is a point in A_k which is not in Λ_n. Either r is in $F_1(i)$ for some $i \geq k$ or r is in L. If r is in $F_1(i)$, for some $i \geq k$, then, since $\overline{F_1(i) \cap \Lambda_n} = F_1(i)$ and $F_1(i) \cap \Lambda_n \subset (A_k \cap \Lambda_n)$, r is in $\overline{A_k \cap \Lambda_n}$. Suppose r is in L. Then there is a sequence $s_1, s_2, s_3, \cdots$ with s_i in $F_1(k+i)$, with limit r. For each positive integer i let t_i be a point of $F_1(k+i) \cap \Lambda_n$ such that $d(s_i, t_i) < 1/i$ (such points exist since $\overline{F_1(k+i) \cap \Lambda_n} = F(k+i)$). Then $t_1, t_2, t_3, \cdots$ is a sequence of points in $\cup_{i \geq k}(F_1(i) \cap \Lambda_n) \subset A_k \cap \Lambda_n$ with limit r. Therefore r is in $\overline{A_k \cap \Lambda_n}$. Hence $\overline{A_k \cap \Lambda_n} = A_k$ for $k = 1, 2, \cdots$.

This proves Lemma 1.

Lemma 2. *Suppose* $n = -1$ *or* $n = 1$, M *is a subcontinuum of* Y, x *is in* Λ_o, x *is a limit point of* $M \cap \Lambda_n$, N *is the component of* $(\overline{M \cap \Lambda_n}) \cap \Lambda_o$ *which contains* x, *and* L *is a subcontinuum of* $\overline{M \cap \Lambda_n}$ *which contains* x *and which intersects* Λ_n. *Then there exists a monotonic sequence* $B_1 \supset B_2 \supset B_3 \supset \cdots$ *of subcontinua of* L *such that*

(1) $\cap_{i>0} B_i = N$, *and*

(2) B_i *intersects* Λ_n *for* $i = 1, 2, 3, \cdots$.

Proof. Let x_1 be a point of $L \cap \Lambda_n$. Let $U_1, U_2, U_3, \cdots$ be a sequence of open subsets of Y closing down on N such that x_1 is not in U_1. For each positive integer i let B_i be the closure of the component of $L \cap U_i$ which contains x. By [6, Theorem 50, p. 18], B_i contains a point which is not in N for $i = 1, 2, 3, \cdots$ and thus B_i contains a point of Λ_n for $i = 1, 2, 3, \cdots$. Since $U_1, U_2, U_3, \cdots$ close down on N, $\cap_{i>0} B_i \subset N$. We note that $N \subset B_i$ for $i = 1, 2, 3, \cdots$ since N is a connected subset of $M \cap U_i$ containing x for all i. Thus $\cap_{i>0} B_i = N$.

This proves Lemma 2.

Lemma 3. *Suppose* $n = -1$ *or* $n = 1$, M *is a subcontinuum of* Y, x *is a point of* Λ_o *which is a limit point of* $M \cap \Lambda_n$

and N *is the component of* $(\overline{M \cap \Lambda_n}) \cap \Lambda_o$ *which contains* x. *Then there is a monotonic sequence* $C_1 \supset C_2 \supset C_3 \supset \cdots$ *of closed subsets of* $\overline{M \cap \Lambda_n}$ *such that*

(1) $\cap_{i>0} C_i \subset N$,

(2) C_i *intersects* Λ_n *for* $i = 1,2,3,\cdots$, *and*

(3) $f(C_i)$ *is connected for* $i = 1,2,3,\cdots$.

Proof. Let L be a subcontinuum of $\overline{M \cap \Lambda_n}$ which contains x and let $A_1, A_2, A_3, \cdots$ be a monotonic sequence of closed subsets of $\overline{M \cap \Lambda_n}$ satisfying (1) through (4) of Lemma 1.

Suppose $L \subset \Lambda_o$. Let $L \subset N$. Define $C_i = A_i$ for $i = 1,2,3,\cdots$. From $L \subset N$ and (1) of Lemma 1, (1) of Lemma 3 follows. Conclusions (2) and (3) of Lemma 3 follow immediately from (2) and (3) of Lemma 1.

If L intersects Λ_n, let $B_1, B_2, B_3, \cdots$ be a monotonic sequence of subcontinua of L satisfying (1) and (2) of Lemma 2, and define $C_i = B_i$ for $i = 1,2,3,\cdots$. Conclusions (1) and (2) follow immediately from (1) and (2) of Lemma 2. Conclusion (3) follows from the fact that B_i is a continuum for $i = 1,2,3,\cdots$.

Theorem 1. *Suppose that* M *is a subcontinuum of* Y, T *is a component of* $M \cap \Lambda_o$ *which contains a limit point of* $M \cap \Lambda_{-1}$ *and a limit point of* $M \cap \Lambda_1$. *Then* $f(T) = H \cap K$.

Proof. Let x be a point of T which is a limit point of $M \cap \Lambda_{-1}$ and let y be a point of T which is a limit point of $M \cap \Lambda_1$. Let N_x be the component of $(\overline{M \cap \Lambda_{-1}}) \cap \Lambda_o$ which contains x and let N_y be the component of $(\overline{M \cap \Lambda_1}) \cap \Lambda_o$ which contains y.

Let $C_x(1), C_x(2), C_x(3), \cdots$, and $C_y(1), C_y(2), C_y(3) \cdots$ be monotonic sequences of closed subsets of $\overline{M \cap \Lambda_{-1}}$ and $\overline{M \cap \Lambda_1}$ respectively such that

(i) $\cap_{i>0} C_x(i) \subset N_x$ and $\cap_{i>0} C_y(i) \subset N_y$,

(ii) $C_x(i)$ intersects Λ_{-1} and $C_y(i)$ intersects Λ_1 for $i = 1,2,3,\cdots$, and

(iii) $f(C_x(i))$ and $f(C_y(i))$ are connected for $i = 1,2,3,\cdots$.

Lemma 3 guarantees the existence of such sequences.

Since N_x and N_y are subsets of $M \cap \Lambda_o$, x is in N_x, y is in N_y, and T is a component of $M \cap \Lambda_o$ containing x and y, it follows that $N_x \subset T$ and $N_y \subset T$. Thus

$$\cap_{i>0}(C_x(i) \cup T \cup C_y(i)) = T.$$

For each positive integer i,

$$f(C_x(i) \cup T \cup C_y(i)) = f(C_x(i)) \cup f(T) \cup f(C_y(i))$$

is a continuum intersecting both $H - (H \cap K)$ and $K - (H \cap K)$. Thus

$$(H \cap K) \subset f(C_x(i)) \cup f(T) \cup f(C_y(i))$$

for $i = 1,2,3,\cdots$ by Proposition 4. Hence

$$\begin{aligned}(H \cap K) &\subset \cap_{i>0}(f(C_x(i) \cup T \cup C_y(i))) \\ &= f(\cap_{i>0}(C_c(i) \cup T \cup C_y(i))) \\ &= f(T).\end{aligned}$$

But, since $T \subset \Lambda_o$, $f(T) \subset H \cap K$. Therefore $f(T) = H \cap K$.

Theorem 2. *Suppose that* Y *is a continuum and* f *is a mapping of* Y *onto* X. *Then there are components* J_H *of* $f^{-1}(H)$ *and* J_K *of* $f^{-1}(K)$ *such that* $f(J_H) = H$, $f(J_K) = K$, *and such that* $f|J_H$ *and* $f|J_K$ *are weakly confluent with respect to* $H \cap K$. *Thus* f *is weakly confluent with respect to* H, K, *and* $H \cap K$.

Proof. We will show the existence of J_H. The existence of J_K follows by essentially the same argument.

Since f maps Y onto $H \cup K$, there is a component of $f^{-1}(H)$ which intersects $f^{-1}(H - (H \cap K)) = \Lambda_{-1}$. Suppose E_1 and E_2 are components of Λ_{-1}. Then, by [6, Theorem 50, p. 18], E_1 and E_2 each have a limit point in Λ_o.

Let $T_1 = f(\overline{E}_1)$ and $T_2 = f(\overline{E}_2)$. Then T_i intersects both $H - (H \cap K)$ and $H \cap K$ for $i = 1,2$. Since X is compact $T_i = \overline{f(E_i)}$ for $i = 1,2$. Thus

$$\begin{aligned} T_i = \overline{f(E_i)} &\subset \overline{f(\overline{E}_i) - (f(\overline{E}_i) \cap K)} \\ &= \overline{T_i - (T_i \cap K)} \subset T_i, \end{aligned}$$

and therefore $\overline{T_i - (T_i \cap K)} = T_i$ for $i = 1,2$. Hence, by Proposition 1, either $T_1 \subset T_2$, or $T_2 \subset T_1$.

We will show that there is a subcontinuum I of Y such that $I \subset \overline{\Lambda}_{-1}$ and $f(I) = \overline{H - (H \cap K)}$.

Let $\{x_1, x_2, x_3, \cdots\}$ be a countable dense subset of $H - (H \cap K)$. For each positive integer i let E(i) be a component of Λ_{-1} such that x_i is in $f(E(i))$, and let $T(i) = f(\overline{E}_i) = \overline{f(E_i)}$. Either

(1) $T(i) = \overline{H - (H \cap K)}$ for some i, or

(2) $T(i)$ is a proper subset of $\overline{H - (H \cap K)}$ for $i = 1,2,3,\cdots$.

If case (1) holds, pick a positive integer i such that $T(i) = \overline{H - (H \cap K)}$ and let $I = T(i)$.

Suppose (2) holds. If i and j are positive integers, then, as noted earlier, either $T(i) \subset T(j)$ or $T(j) \subset T(i)$. Hence there is a monotonic subsequence $T_1(1) \subset T_1(2) \subset T_1(3) \subset \cdots$ of $T(1), T(2), T(3), \cdots$. Let $E_1(1), E_1(2), E_1(3), \cdots$ be the corresponding subsequence of $E(1), E(2), E(3), \cdots$.

Since Y is compact, there is a subsequence $\overline{E_2(1)}, \overline{E_2(2)}, \overline{E_2(3)}, \cdots$ of $\overline{E_1(1)}, \overline{E_1(2)}, \overline{E_1(3)}, \cdots$ which has a sequential limiting set I which is a continuum [6, Theorems 58, 59, pp. 23-24]. Let $T_2(i) = \overline{E_2(i)}$ for $i = 1,2,3,\cdots$. Now

$$f(I) = \overline{\cup_{k>0} f(E_2(k))} = \overline{\cup_{k>0} T_2(k)} = \overline{\cup_{k>0} T(k)}.$$

Thus $\overline{H - (H \cap K)} = \overline{\{x_1, x_2, x_3, \cdots\}} \subset f(I)$. Since $I \subset \overline{\Lambda}_{-1}$, $f(I) \subset \overline{H - (H \cap K)}$, therefore $f(I) = \overline{H - (H \cap K)}$.

Thus, in either case we have shown the existence of the continuum I.

Let J_H be the component of $f^{-1}(H)$ which contains I. From [6, Theorem 52a, p. 21], J_H contains a point, r, which is a limit point of Λ_1. Let L be the component of $J_H \cap \Lambda_o$ which contains r. Then, since J_H intersects Λ_{-1}, L contains a point which is a limit point of $J_H \cap \Lambda_{-1}$, again by [6, Theorem 52a]. Thus, by Theorem 1, $f(L) = H \cap K$. Hence

$$H = (H \cap K) \cup (H - (H \cap K)) \subset f(L) \cup f(I) \subset f(J_H).$$

But $J_H \subset f^{-1}(H)$ so $f(J_H) = H$.

This concludes the proof of Theorem 2.

4. Continua Decomposable Into Continua in Class W

We now use Theorem 2 to prove the following sum theorem for continua in class W.

Theorem 3. *Suppose* H *and* K *are continua in class* W *which intersect,* $H \cap K$ *is connected, and* $H \cup K$ *is atriodic. Then* $H \cup K$ *is in class* W.

Proof. Suppose Y is a continuum and f is a mapping of Y onto $H \cup K$. If $H \subset K$ or $K \subset H$ the conclusion follows trivially. Thus suppose that H is not a subset of K and that K is not a subset of H. We adopt the notation of the previous section: $\Lambda_{-1} = f^{-1}(H - (H \cap K))$, $\Lambda_o = f^{-1}(H \cap K)$, and $\Lambda_1 = f^{-1}(K - (H \cap K))$.

Suppose C is a subcontinuum of $H \cup K$.

Case I. Suppose $C \subset H$. Let J_H be a component of $f^{-1}(H)$ such that $f(J_H) = H$ (Theorem 2). Let g be the restriction of f to J_H. Then g is a mapping of J_H onto H and since H is in class W there is a component L of $g^{-1}(C)$ such that $g(L) = f(L) = C$. Since J_H is a component of $f^{-1}(H)$, L is a component of $g^{-1}(C)$, and $C \subset H$, L is a component of $f^{-1}(C)$.

Case II. Suppose $C \subset K$. The proof for this case is identical with that for Case I.

Case III. Suppose C intersects each of $H - (H \cap K)$ and $K - (H \cap K)$.

Subcase III-a. Suppose, in addition, that neither H nor K is a subset of C.

Since C intersects $H - (H \cap K)$ and $K - (H \cap K)$, $H \cap K \subset C$ by Proposition 4. Let $H' = H \cup C$ and $K' = K \cup C$; H' and K' are continua. Moreover

$$H' \cap K' = (H \cup C) \cap (K \cup C)$$
$$= (H \cap K) \cup C = C$$

which is a continuum, and $H' \cup K' = H \cup K$ is atriodic. Thus, by Theorem 2, f is weakly confluent with respect to $H' \cap K' = C$.

Subcase III-b. Suppose, in addition to the supposition of Case III, that $H \subset C$.

If $K \subset C$, then $C = H \cup K$ and $f(Y) = C$, so suppose that K is not a subset of C. From Proposition 2 it follows that $C - H$ is connected.

Let $H' = C$ and $K' = K$. Then

$$\begin{aligned} H' \cap K' = C \cap K &= [(C \cap H) \cup (C - H)] \cap K \\ &= [H \cup (C - H)] \cap K \\ &= (H \cap K) \cup [(C - H) \cap K] \\ &= (H \cap K) \cup (C - H) \\ &= (H \cap K) \cup (\overline{C - H}). \end{aligned}$$

Now $\overline{C - H}$ intersects $H \cap K$ by [6, Theorem 50, p. 18], and thus $H' \cap K'$ is the union of two continua with a point in common and is therefore a continuum [6, Theorem 30, p. 11]. Also $H' \cup K' = H \cup K$ is atriodic. Therefore, by Theorem 2, f is weakly confluent with respect to $H' = C$.

Subcase III-c. Suppose, in addition to the supposition of Case III, that $K \subset C$. The argument for this subcase is identical with the argument for Subcase III-b.

This concludes the proof of Theorem 3.

Continua in class W are unicoherent. This fact was first observed in the Topology Seminar at the University of Houston as recorded in S. B. Nadler's book [7, p. 501] and in the paper by Grispolakis, Nadler and Tymchatyn [3] where it was established to be a corollary of two other results obtained in that paper. For the sake of completeness we include the following proof of this fact before using it to reformulate Theorem 3.

Theorem 4. *Continua in class* W *are unicoherent.*

Proof. Suppose that M is a continuum which is not unicoherent. Then there are subcontinua H and K such that $M = H \cup K$ and such that $H \cap K$ is the union of two mutually exclusive closed point sets A and B.

Let p be a point in A and q be a point in B. Let

$$X = (H \times \{0\}) \cup (\{q\} \times [0,1]) \cup (K \times \{1\}),$$

and let f be the restriction of π_1 to X (π_1 is the first projection of $M \times [0,1]$ onto M).

Let U be an open set containing A whose closure does not intersect B. Let T_H and T_K be the closures of the components of $U \cap H$, and $U \cap K$, respectively, which contain p. By [6, Theorem 50, p. 18] T_H and T_K contain points r_H and r_K, respectively, which are not in U. Since $\bar{U}$ and B are mutually exclusive it follows that r_H is not in K and r_K is not in H.

Let $C = T_H \cup T_K$. This set is a subcontinuum of M. Then $f^{-1}(C)$ has just two components, $T_H \times \{0\}$ and $T_K \times \{1\}$. However r_H is not in $f(T_K \times \{1\})$ and r_K is not in $f(T_H \times \{0\})$. Thus f is not weakly confluent.

Therefore M is not in class W.

We now reformulate Theorem 3 along the lines of [4, Theorem 2, p. 196].

Theorem 5. *Suppose* H *and* K *are continua in class* W *which intersect and the* $H \cup K$ *is atriodic. Then* $H \cup K$ *is in class* W *if and only if* $H \cap K$ *is connected.*

5. Upper Semicontinuous Decompositions Into Continua in Class W

Suppose that M is a continuum. A collection $\mathcal{D}$ of mutually exclusive closed subsets of M whose union is M is said to be an *upper semicontinuous decomposition* of M provided that for each element D in $\mathcal{D}$ and each open subset U of M such that $D \subset U$, there is an open subset V of M such that each element of $\mathcal{D}$ which intersects V is a subset of U. If $\mathcal{D}$ is an upper semicontinuous decomposition of M, the function $\eta: M \to \mathcal{D}$ such that $\eta(x)$ is the element of $\mathcal{D}$ to which x belongs, is called the *natural map* or *natural projection* induced by $\mathcal{D}$. Treating the members of $\mathcal{D}$ as points, we define a topology on $\mathcal{D}$ by defining a subset U of $\mathcal{D}$ to be open provided $\eta^{-1}(U)$ is open in M. The collection $\mathcal{D}$ equipped with this topology is denoted by $M/\mathcal{D}$ and called the *upper semicontinuous decomposition space* determined by $\mathcal{D}$. If each element of $\mathcal{D}$ is connected (equivalently, η is a monotone map) $M/\mathcal{D}$ is called a *monotone upper semicontinuous decomposition* of M.

C. Wayne Proctor [8] has proved that if the continuum M has an upper semicontinuous decomposition $\mathcal{D}$, each element of which is a C-set and is in class W, such that $M/\mathcal{D}$ is in class W, then M is in class W. In the following theorem we reach the same conclusion from somewhat different hypotheses.

Theorem 6. *Suppose that* M *is an atriodic continuum,* $\mathcal{D}$ *is a monotone upper semicontinuous decomposition of* M, *each element of* $\mathcal{D}$ *is in class* W, *and* $M/\mathcal{D}$ *is an arc. Then* M *is in class* W.

Proof. Let $\eta: M \to [0,1]$ be a monotone map such that $\mathcal{D} = \{\eta^{-1}(t): t \in [0,1]\}$. Suppose that X is a continuum and f is a mapping of X onto M.

We first note that if C separates M then f is weakly confluent with respect to C. Since M is atriodic M - C has just two components. Let H denote the union of C and one of these components and let K denote the union of C and the other. Then H and K are continua, $H \cap K = C$ is a continuum and $H \cup K = M$ is atroidic. From Theorem 2 it follows that f is weakly confluent with respect to C.

We now show that if $0 < t < 1$, then f is weakly confluent with respect to $\eta^{-1}([0,t])$ and $\eta^{-1}([t,1])$. To see this let $H = \eta^{-1}([0,t])$ and $K = \eta^{-1}([t,1])$. Then $H \cap K = \eta^{-1}(t)$ is a continuum and, again appealing to Theorem 2, f is weakly confluent with respect to both H and K.

We next show that f is weakly confluent with respect to $\eta^{-1}(t)$ for all t in [0,1]. If $0 < t < 1$ then $\eta^{-1}(t)$ separates M and the conclusion follows from the above observation. Suppose $t = 0$. In [1, Lemma 2] it is shown that if f is weakly confluent with respect to each member of a monotonic collection of subcontinua of M then f is weakly confluent with respect to the intersection of the members of the collection. The collection $\mathcal{G} = \{\eta^{-1}([0,t]): 0 \leq t \leq 1\}$ is such a collection and $\cap \mathcal{G} = \eta^{-1}(0)$. Thus f is weakly confluent with respect to $\eta^{-1}(0)$. Likewise f is weakly confluent with respect to $\eta^{-1}(1)$.

Suppose that C is a subcontinuum of M which does not separate M.

Case I. Suppose that $C \subset \eta^{-1}(t)$ for some t in [0,1]. Now f is weakly confluent with respect to $\eta^{-1}(t)$ so there is a component L of $f^{-1}(\eta^{-1}(t))$ such that $f(L) = \eta^{-1}(t)$. Since $\eta^{-1}(t)$ is in class W $f|L$ is weakly confluent with respect to C, and hence f is weakly confluent with respect to C.

Case II. Suppose that C is not a subset of any member of $\mathcal{D}$. Then $\eta(C) = [a,b]$, $0 \leq a < b \leq 1$. We claim that either $\eta^{-1}(0) \subset C$ or $\eta^{-1}(1) \subset C$. Suppose that this does not hold, and there are points p in $\eta^{-1}(0)$ and q in $\eta^{-1}(1)$ which do not belong to C. Let $a < t < b$. Let $H = \eta^{-1}([0,t])$ and $K = \eta^{-1}([t,1])$. Since $H \cup K = M$ is atroidic $H \cap K = \eta^{-1}(t)$ is a continuum and C intersects both $H - (H \cap K)$ and $K - (H \cap K)$, it follows from Proposition 4 that $\eta^{-1}(t) \subset C$. Since $\eta^{-1}(t)$ separates p from q, and since $\eta^{-1}(t) \subset C$, so does C. This is inconsistent with the supposition that C does not separate M, so the claim holds. Suppose that $\eta^{-1}(0) \subset C$. Then $\eta(C) = [0,b]$, $b \leq 1$. We have seen above that f is weakly confluent with respect to $\eta^{-1}([0,b])$. Let L be a component of $f^{-1}(\eta^{-1}([0,b]))$ such that $f(L) = \eta^{-1}([0,b])$. If $C = \eta^{-1}([0,b])$ we are done. Suppose that $C \neq \eta^{-1}([0,b])$. Using an argument similar to that presented above, we have that $\eta^{-1}(t) \subset C$ for $0 < t < b$. Since $\eta^{-1}(0) \subset C$ it follows that there is a point q in $\eta^{-1}(b)$ such that q is not in C. Suppose $0 < t < b$. Then $\eta^{-1}([t,b)) \subset C$ and thus

$$C \cap \eta^{-1}([t,b]) = \eta^{-1}([t,b)) \cup (C \cap \eta^{-1}(b))$$

is a continuum, since every component of $C \cap \eta^{-1}(b)$ contains a limit point of $C - (C \cap \eta^{-1}(b))$ and thus of $\eta^{-1}([t,b))$. Since $(C \cap \eta^{-1}([t,b])) \cup \eta^{-1}(b) = \eta^{-1}([t,b])$ is a proper subcontinuum of M with interior, it is unicoherent [2, Theorem 1.5]. Hence $C \cap \eta^{-1}(b) = (C \cap \eta^{-1}([t,b])) \cap \eta^{-1}(b)$ is connected. Letting $H = C$ and $K = \eta^{-1}(b)$ we see from Theorem 2 that $f|L$ is weakly confluent with respect to C and thus f is weakly confluent with respect to C.

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