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Let κ be an infinite cardinal. A collection of subsets of a set is said to be *Noetherian* (κ-*Noetherian*) provided every subcollection well ordered set inclusion is finite (has at most κ elements). Clearly, as in ring theory, a collection of sets is Noetherian if and only if every subcollection contains a maximal element. The concept of a Noetherian base for a topological space was introduced in [N_1], [GN], and [LN]. The concept of an ω-Noetherian collection of sets was introduced in [N_3] where it is shown, for example, that a topological space $(X,\mathcal{J})$ is hereditarily Lindelöf if and only if $\mathcal{J}$ is ω-Noetherian.

It is not difficult to show that for any topological space X and any $x \in X$ every neighborhood base for x contains a Noetherian neighborhood base for x (see [dG] for the analogous result for covers). This fact was used in [Gr] to show that a topological space is globular [Sc] if and only if every point has a neighborhood base with subinfinite rank. Clearly, a T_0-space need not have a Noetherian base. However, to the best of my knowledge, the following is the only result concerning the existence of T_1-spaces without Noetherian bases.

Theorem 0.1 [vD]. *If* $\alpha \in$ Ord *then* α *with the order topology has a Noetherian base if and only if* $\alpha < \kappa$ *where* κ *is the first strongly inaccessible cardinal.*

Since Con(ZFC) → Con(ZFC + there are no strongly inaccessible cardinals), see [Je], no "real" T_1-space has been shown to not have a Noetherian base.

A collection of subsets of a set is said to have *subinfinite (countable) rank* provided every infinite (uncountable) subcollection with nonempty intersection has two members related by set inclusion. The concept of a base with subinfinite (countable) rank was introduced in [GN] and studied further in [FG], [G], [LN], [N_1], [N_2], [N_3] among others.

Although a Noetherian base appears to be a very weak base property, spaces having Noetherian bases with subinfinite rank have been shown to possess many interesting properties, see [GN] and [N_4]. In this paper it is observed that a weakly uniform base is Noetherian. This fact is used to show that, although a weakly uniform base need not be point countable, a weakly uniform base with subinfinite rank is σ-point finite (Theorem 1.6). We also show that, even though the perfect image of a space having a Noetherian base need not have a Noetherian base (Theorem 3.5), the perfect image of a space with a Noetherian base with subinfinite rank has an ω-Noetherian base (Proposition 3.6).

Noetherian (κ-Noetherian) collections can be used to characterize certain covering properties. For example in this paper T_1 meta-Lindelöf spaces are characterized as those T_1-spaces for which every open cover has a Noetherian open refinement with countable rank (Theorem 2.4). We also use certain ω-Noetherian collections to characterize paracompact Go-spaces (Corollary 2.8).

We show that various classes of T_1-spaces, for example those with point countable bases, have Noetherian bases (Theorem 1.2 and Corollary 1.3). It is also shown that the product of spaces with κ-Noetherian bases has a κ-Noetherian base (Theorem 3.1) and the uncountable product of nontrivial T_1-spaces never has a base of countable rank (Theorem 3.3).

We will use Greek letters to denote ordinals and for convenience will not distinguish between the cardinal κ and the first ordinal having cardinality κ. The first infinite cardinal will be denoted by ω. For any set A the cardinality of A will be denoted by $|A|$. If $\mathcal{H}$ is a collection of subsets of a set X and $x \in X$ then $(\mathcal{H})_x = \{H \in \mathcal{H}: x \in H\}$. When we say that a collection of sets is κ-Noetherian it is understood that κ is an infinite cardinal.

1. Spaces Having Special Noetherian Bases

Suppose a topological space X has a base with some property. It is natural to ask if X has a Noetherian base with the same property [LN]. In general, the answer is no. For example, ω_1 with the order topology has a Noetherian base and a clopen base of intervals. However, Brian Scott has shown that it does not have a Noetherian clopen base and in [LN] it is observed that it does not have a Noetherian base of intervals. Also, the Sorgenfrey line has a Noetherian base and a base of subinfinite rank but does not have a Noetherian base of subinfinite rank.

Theorem 1.1 [F]. *A T_1-space with a base with (point-) finite rank has a Noetherian base with (point-)finite rank.*

Using Theorem 1.1 one can greatly simplify the proofs of many theorems concerning spaces with bases of point-finite rank.

Theorem 1.2. *A* T_1*-space with a point countable base has a Noetherian point countable base.*

Proof. Let X be a T_1-space having a point countable base $\mathcal{B}$. Let $S = \{x \in X\colon x \text{ is isolated}\}$ and $\mathcal{S} = \{\{x\}\colon x \in S\}$. Well order $X\setminus S$, say $X\setminus S = \{x(\alpha)\colon \alpha < \kappa\}$ and for every $\alpha < \kappa$ let $\mathcal{B}(\alpha) = \{B \in \mathcal{B}\colon \alpha = \min\{\gamma < \kappa\colon x(\gamma) \in B\}\}$. For $\alpha < \kappa$ with $|\mathcal{B}(\alpha)| < \omega$ let $\mathcal{B}'(\alpha) = \mathcal{B}(\alpha)$.

Suppose $\alpha < \kappa$ and $|\mathcal{B}(\alpha)| = \omega$. Let $\mathcal{B}(\alpha) = \{B(\alpha,n)\colon n < \omega\}$, $B'(\alpha,0) = B(\alpha,0)$ and choose $y(\alpha,0) \in B(\alpha,0)\setminus\{x(\alpha)\}$. Suppose for all $m < n$, $B'(\alpha,m) \subseteq B(\alpha,m)$ has been defined and $y(\alpha,m) \in B'(\alpha,m)\setminus\{x(\alpha)\}$ has been chosen. Let $B'(\alpha,n) = B(\alpha,n)\setminus\{y(\alpha,m)\colon m < n \text{ and } B'(\alpha,m) \subseteq B(\alpha,n)\}$. Let $\mathcal{B}'(\alpha) = \{B'(\alpha,n)\colon n < \omega\}$ and $\mathcal{B}' = \mathcal{S} \cup (\cup\{\mathcal{B}'(\alpha)\colon \alpha < \kappa\})$.

Suppose for every $m < \omega$, $\alpha(m) < \kappa$ and $n(m) < \omega$ have been chosen such that if $m < k < \omega$ then $B'(\alpha(m),n(m)) \subseteq B'(\alpha(k),n(k))$. If $\alpha < \beta < \kappa$ then $x(\alpha) \notin \cup\mathcal{B}(\beta)$ and so $x(\alpha) \notin \cup\mathcal{B}'(\beta)$. Thus if $m < k < \omega$ then $\alpha(m) \geq \alpha(k)$ and so there is an $n < \omega$ such that if $n < m < \omega$ then $\alpha(n) = \alpha(m)$. Hence to show that $\mathcal{B}'$ is Noetherian we need only show that for each $\alpha < \kappa$, $\mathcal{B}'(\alpha)$ is Noetherian.

Suppose $\alpha < \kappa$ and $|\mathcal{B}(\alpha)| = \omega$ and for all $m < \omega$, $n(m) < \omega$ has been chosen. There is an infinite $A \subseteq \omega$ such that if $k,m \in A$ with $k < m$ then $n(k) \leq n(m)$. If $k,m \in A$ with $k < m$ then $B'(\alpha,n(k)) \not\subset B'(\alpha,n(m))$. Thus $\mathcal{B}'(\alpha)$ does not contain an infinite well ordered increasing subset, i.e. it is Noetherian.

Since $\mathcal{B}'$ is clearly point countable all that remains to be shown is that it is a base. Let $x \in X \setminus S$ and V an open neighborhood of x. There is an $\alpha < \kappa$ and an $n < \omega$ such that $x \in B(\alpha,n) \subseteq V$. If there is an $m < n$ with $x \in B'(\alpha,m) \subseteq B(\alpha,n)$ then we are done. Thus suppose that if $m < n$ and $B'(\alpha,m) \subseteq B(\alpha,n)$ then $x \notin B'(\alpha,m)$. Then by the definition of $B'(\alpha,n)$, $x \in B'(\alpha,n) \subseteq B(\alpha,n) \subseteq V$. Thus $\mathcal{B}'$ is a base for X.

Corollary 1.3. *If a* T_1*-space has a base which is any of the following then it has a Noetherian base with the same property:*

(1) *σ-point finite*

(2) *σ-disjoint*

(3) *σ-discrete*

(4) *σ-locally finite*

(5) *σ-locally countable*

(6) *locally countable.*

Proof. If the base $\mathcal{B}$ in the proof of Theorem 1.2 satisfies any of the conditions (1)-(6) then so does $\mathcal{B}'$.

A base $\mathcal{B}$ for a topological space X is called a *(weakly) uniform base* provided if $x \in X$ and $\mathcal{G}$ is any infinite subset of $(\mathcal{B})_x$ then $\mathcal{G}$ is a neighborhood base at x ($\cap\mathcal{G} = \{x\}$). The concept of a uniform base was introduced in [A] and weakly uniform base was introduced in [HL].

Theorem 1.4 [F]. *A base for a topological space is a uniform base if and only if it is a Noetherian base of countable order with subinfinite rank.*

The following is easily proved.

Proposition 1.5. *A weakly uniform base for a topological space is Noetherian.*

A space having a weakly uniform base need not have a base with subinfinite rank (see Theorem 3 [DRW]) nor a base of countable order (the Michael line, Example 71 of [SS]). In [DRW] under the assumption of Martin's Axiom and $\omega_2 < 2^\omega$, they construct a first countable T_1-space with a weakly uniform base which does not have a point countable base. For spaces having weakly uniform bases with subinfinite rank the situation is different.

Proposition 1.6. *If $\mathcal{B}$ is a weakly uniform base with subinfinite rank for a topological space* X *then $\mathcal{B}$ is σ-point finite.*

Proof. Let $\mathcal{B}(0)$ be the set of all maximal elements in the partially ordered set $(\mathcal{B},\subseteq)$. Notice that if $B \in \mathcal{B}\backslash\mathcal{B}(0)$ then there is a $B' \in \mathcal{B}(0)$ such that $B \subset B'$. Also, since $\mathcal{B}$ is Noetherian and has subinfinite rank the collection $\mathcal{B}(0)$ is point finite. Suppose for $m \leq n < \omega$, $\mathcal{B}(m) \subseteq \mathcal{B}$ has been chosen. Let $\mathcal{B}(n+1)$ be the set of all maximal elements in the partially ordered set $(\mathcal{B}\backslash\cup\{\mathcal{B}(m): m \leq n\},\subseteq)$. Since $\mathcal{B}\backslash\cup\{\mathcal{B}(m): m \leq n\}$ is Noetherian and has subinfinite rank $\mathcal{B}(n+1)$ is point finite and if $B \in \mathcal{B}\backslash\cup\{\mathcal{B}(m): m \leq n+1\}$ then there is a $B' \in \mathcal{B}(n+1)$ such that $B \subset B'$.

Let $\mathcal{B}(\omega) = \{\{x\}: x \in X, \{x\} \in \mathcal{B}\backslash\cup\{\mathcal{B}(n): n < \omega\}\}$. Clearly $\cup\{\mathcal{B}(\alpha): \alpha \leq \omega\}$ is σ-point finite. Suppose $B \in \mathcal{B}\backslash\cup\{\mathcal{B}(n): n < \omega\}$. Then for every $n < \omega$ there is a

$B(n) \in \mathcal{B}(n)$ such that $B \subset B(n)$. Since, for $n,m < \omega$ with $m \neq n$, $\mathcal{B}(n) \cap \mathcal{B}(m) = \phi$, the collection $\{B(n): n < \omega\}$ is a countable subset of $\mathcal{B}$ and $B \subset \cap\{B(n): n < \omega\}$. Hence, since $\mathcal{B}$ is a weakly uniform base, $|B| = 1$ and so $B \in \mathcal{B}(\omega)$. Thus $\mathcal{B} = \cup\{\mathcal{B}(\alpha): \alpha \leq \omega\}$.

A T_2-space with a σ-point finite base (and hence, by Corollary 1.3, a Noetherian σ-point finite base) need not have a base with subinfinite rank (Example 1 of [BL]). The Michael line is a space with a weakly uniform base with subinfinite rank which does not have a uniform base.

A base $\mathcal{B}$ for a topological space X is called a *(weak) base of countable order* provided if $\{B(n): n < \omega\} \subset \mathcal{B}$ such that $n < m < \omega$ implies $B(n) \supset B(m)$ and $x \in \cap\{B(n): n < \omega\}$ then the collection $\{B(n): n < \omega\}$ is a neighborhood base at x $(\cap\{B(n): n < \omega\} = \{x\})$. The concept of a base of countable order was introduced in [Ar]. The following is the natural analog of Theorem 1.4. It follows directly from Proposition 1.5 and Lemma 3.6.

Theorem 1.7. *A base for a topological space is a weakly uniform base with subinfinite rank if and only if it is a Noetherian weak base of countable order with subinfinite rank.*

In Theorem 1.7 the subinfinite rank condition is needed, since ω_1 with the order topology has a Noetherian base of countable order but does not have a weakly uniform base.

Suppose κ is a strongly inaccessible cardinal. By Theorem 0.1, κ with the order topology does not have a

Noetherian base. In fact, if S is any stationary subset of κ then S with the subspace topology does not have a Noetherian base. Thus $\{\alpha < \kappa: cf(\alpha) \leq \omega\}$ has a base of countable order but does not have a Noetherian base. It is not known if a T_1 developable space must have a Noetherian base (and therefore a Noetherian development).

2. Covering Properties

A cover $\mathcal{G}$ of a set X is called *minimal* provided no proper subcollection of $\mathcal{G}$ covers X. A topological space is called *irreducible* provided every open cover has a minimal open refinement. Clearly a minimal cover of a set X is Noetherian. However, ω_1 with the order topology has a Noetherian base but is not irreducible. If $\mathcal{G}$ is a Noetherian cover of a set X then the subcover $\mathcal{H}$ consisting of all maximal elements of $(\mathcal{G},\subseteq)$ has the property that for every $H,H' \in \mathcal{H}$, $H \not\subset H'$ and $H' \not\subset H$. This subcover shows that a Noetherian cover is a natural generalization of a minimal cover.

Wicke and Worrell observed that θ-refinable spaces are irreducible [WW]. Although weakly θ-refinable spaces need not be irreducible (see [BL] and [vDW]), in [S] it is shown that weakly $\overline{\theta}$-refinable spaces are irreducible. It is not known if T_1 weakly $\overline{\delta\theta}$-refinable (or even meta-Lindelöf) spaces are irreducible. It is also not known if every open cover of a T_1 weakly θ-refinable space has a Noetherian open refinement.

The following lemma is proved in the same way as Theorem 1.2.

Lemma 2.1. *Suppose* $\mathcal{G}$ *is a collection of open subsets of a space* X *and* $A \subseteq \cup\mathcal{G}$ *such that for every* $G \in \mathcal{G}$, $A \cap G \neq \phi$ *and for every* $x \in A$, $|(\mathcal{G})_x| \leq \omega$. *Then for every* $G \in \mathcal{G}$ *there is an open* $B(G) \subseteq G$ *such that* $\cup\{B(G): G \in \mathcal{G}\} = \cup\mathcal{G}$ *and* $\{B(G): G \in \mathcal{G}\}$ *is Noetherian.*

Proposition 2.2. *If* X *is a* T_1 *weakly* $\overline{\delta\theta}$*-refinable space then every open cover of* X *has an open Noetherian weak* $\overline{\delta\theta}$*-refinement.*

Proof. Let $\mathcal{G} = \cup\{\mathcal{G}(n): n < \omega\}$ be an open cover of X satisfying the following conditions:

(i) for each $x \in X$ there is an $n(x) < \omega$ such that $0 < |(\mathcal{G}(n(x)))_x| \leq \omega$

(ii) $\{\cup\mathcal{G}(n): n < \omega\}$ is point finite,

i.e. $\mathcal{G}$ is a weak $\overline{\delta\theta}$-cover. For each $m < \omega$ let $A(m) = \{x \in X: m = n(x)\}$ and $\mathcal{H}(m) = \{G \in \mathcal{G}(m): G \cap A(m) \neq \phi\}$.

By Lemma 2.1 for each $n < \omega$ and each $H \in \mathcal{H}(n)$ there is an open $W(H) \subseteq H$ such that $\cup\{W(H): H \in \mathcal{H}(n)\} = \cup\mathcal{H}(n)$ and $\{W(H): H \in \mathcal{H}(n)\}$ is Noetherian. For each $n < \omega$ let $\mathcal{W}(n) = \{W(H): H \in \mathcal{H}(n)\}$. By (ii) $\{\cup\mathcal{W}(n): n < \omega\}$ is point finite and so $\mathcal{W} = \cup\{\mathcal{W}(n): n < \omega\}$ is Noetherian. Since conditions (i) and (ii) hold for $\mathcal{W}$, $\mathcal{W}$ is a Noetherian weak $\overline{\delta\theta}$-refinement of $\mathcal{G}$.

When subinfinite rank is introduced things become clear.

Theorem 2.3 [GN]. *A topological space is metacompact if and only if every open cover has a Noetherian open refinement with subinfinite rank.*

The following analog of the above theorem follows from Lemma 2.1.

Theorem 2.4. *A* T_1*-space is metaLindelöf if and only if every open cover has a Noetherian open refinement with countable rank.*

Theorem 2.5 [FG]. *A* T_3*-space is metacompact if and only if every open cover has an* ω*-Noetherian open refinement with subinfinite rank.*

It is not known if the metaLindelöf analog of Theorem 2.5 holds.

For generalized ordered spaces (Go-spaces) we can use certain ω-Noetherian collections to characterize paracompactness. First we state the following well known characterization of paracompact Go-spaces.

Theorem 2.6 [EL]. *The following are equivalent for a generalized ordered space* X*:*

(a) X *is not paracompact*

(b) *for some ordinal* λ *with* $\mathrm{cf}(\lambda) = \lambda > \omega$*, some stationary subset of* λ *is homeomorphic to a closed subspace of* X.

Lemma 2.7. *Suppose* κ *is an ordinal with* $\mathrm{cf}(\kappa) = \kappa > \omega$ *and* S *is a stationary subset of* κ*. The open cover* $\mathcal{U} = \{[0,\alpha] \cap S : \alpha \in S\}$ *of* S *does not have an* ω*-Noetherian open refinement consisting of intervals in* S.

Proof. Let $\mathcal{V}$ be an open refinement of $\mathcal{U}$ consisting of intervals in S. For every $V \in \mathcal{V}$ let $\gamma(V) \in S$ and

$\alpha(V), \beta(V) \in \kappa$ such that $(\alpha(V), \beta(V)) \cap S = V$ and $V \subseteq [0, \gamma(V)]$. For every $\sigma \in S \cap \text{Lim}$ let $V(\sigma) \in (\mathcal{V})_\sigma$, $\alpha(\sigma) = \alpha(V(\sigma))$, $\beta(\sigma) = \beta(V(\sigma))$ and $\gamma(\sigma) = \gamma(V(\sigma))$. Notice that for all $\sigma \in S \cap \text{Lim}$, $\alpha(\sigma) < \sigma$.

Since S is a stationary subset of κ, so is $S \cap \text{Lim}$. Thus, by the "Pressing Down Lemma" (see [Fl] 2.2), there is a $\lambda \in \kappa$ and an $A \subseteq S \cap \text{Lim}$ with $|A| = \kappa$ such that for all $\nu \in A$, $\alpha(\nu) = \lambda$. Since $|A| = \kappa$, $\sup(A) = \kappa$. Also for $\nu \in A$, since $V(\nu) \subseteq [0, \gamma(\nu)]$ and $\gamma(\nu) < \kappa$, $|V(\nu)| < \kappa$. Thus there is an $A' \subseteq A$ with $|A'| = \kappa$ such that if $\nu, \nu' \in A'$ and $\nu < \nu'$ then $V(\nu) \subset V(\nu')$. Thus $\mathcal{V}$ is not ω-Noetherian. In fact, for any infinite cardinal $\mu < \kappa$, $\mathcal{V}$ is not μ-Noetherian.

Corollary 2.8. *A generalized ordered space* X *is paracompact if and only if every closed subspace* H *has the property that every relatively open cover of* H *has an* ω-*Noetherian refinement consisting of relatively open intervals in* H.

In Corollary 2.8 we cannot avoid looking at closed subspaces. The following example is a non-paracompact Go-space having a Noetherian base of intervals.

Example 2.9. Let X be constructed from ω_1 by placing between each ordinal α and its successor $\alpha+1$ a copy of ω_1 (say $\omega_1 \times \{\alpha\}$). Topologize X with the obvious order topology and let X* be the Go-space obtained from X by isolating all elements of X except the limit ordinals of the original copy of ω_1. Since ω_1 is a closed subspace of

X^*, by Theorem 2.6, X^* is not paracompact. The collection $\{\{x\}: x \in X\backslash(\omega_1 \cap \mathrm{Lim})\} \cup \{((\beta,\alpha),\beta]: \alpha < \beta \text{ and } \beta \in \omega_1 \cap \mathrm{Lim}\}$ is a Noetherian base for X^* consisting of intervals.

3. Products and Perfect Images of Spaces Having Noether Bases

In [LN] it is noted that the product of spaces having Noetherian bases has a Noetherian base. This is also true for products of spaces having κ-Noetherian bases.

Theorem 3.1. *Let* κ *be an infinite cardinal,* A *a non-empty set and for every* $a \in A$ *let* X_a *be a topological space having a* κ*-Noetherian base. The space* $X = \Pi_{a \in A} X_a$ *has a* κ*-Noetherian base.*

Theorem 3.2 [GN]. *The finite product of spaces having Noetherian bases of subinfintie rank has a Noetherian base of subinfinite rank.*

The Sorgenfrey line S has an ω-Noetherian base with subinfinite rank. However S^2 does not have a base of countable rank [LN].

A collection of sets is said to be *well ranked* provided it is the countable union of Noetherian subcollections with subinfinite rank. In [GN] it is shown that the countable product of spaces with well ranked bases has a well ranked base. It is not known if the countable product of spaces with Noetherian bases of subinfinite rank must have a Noetherian base of subinfinite rank. However such products do have Noetherian well ranked bases. It follows from the next proposition that the uncountable product of nontrivial T_1-spaces can never have a well ranked base.

Proposition 3.3. *Suppose* X *is the product of uncountably many* T_1*-spaces each having at least* 2 *points. Then no point of* X *has a neighborhood base with countable rank.*

Proof. Since having a neighborhood base of countable rank is hereditary, it suffices to show:

$$\left\{\begin{array}{l}\text{If } X = \Pi_{\alpha<\omega_1}\{0,1\}_\alpha \text{ where for all } \alpha < \omega_1,\ \{0,1\}_\alpha \text{ is} \\ \{0,1\} \text{ with the discrete topology, } f = \langle 0: \alpha < \omega_1 \rangle, \\ \text{and } \mathcal{B} \text{ is a neighborhood base for f then } \mathcal{B} \text{ does not} \\ \text{have countable rank.}\end{array}\right.$$

For all $\alpha < \omega_1$ let $B(\alpha) \in \mathcal{B}$ such that $B(\alpha) \subseteq \pi_\alpha^{-1}(\{0\})$.

It is a straightforward application of the "delta system lemma" (see [J] A2.2) to find an uncountable incomparable subset of $\{B(\alpha): \alpha < \omega_1\}$.

A mapping f from a topological space X onto a topological space Y is said to be *perfect* provided f is closed, continuous and for every $y \in Y$, $f^{-1}(y)$ is compact. Dennis Burke has observed that the construction in Lemma 1 and Theorem 2 of $[B_2]$ gives a Noetherian base. Thus Corollary 5 of $[B_2]$ can be restated as follows:

Theorem 3.4 (Burke). *A Noetherian base is not necessarily preserved under a perfect map. In fact, if* Y *is any space then* Y *is the image, under an open perfect mapping, of a space with a Noetherian base.*

It is not known if the perfect image of a space having a Noetherian base with subinfinite rank must have a Noetherian base with subinfinite rank or even a Noetherian base. However the following proposition does imply that the perfect image of a space with a Noetherian base with subinfinite rank does have an ω-Noetherian base.

The following lemma follows from $\omega \to (\omega)^2_3$, (see [J] A4).

Lemma 3.5. *Suppose* X *is a set,* A *is an infinite subset of* ω *and for each* $n \in A$, $S(n) \subseteq X$ *such that* $\{S(n): n \in A\}$ *is Noetherian, has subinfinite rank and* $\cap\{S(n): n \in A\} \neq \phi$. *Then there is an* $A' \subseteq A$ *with* $|A'| = \omega$ *such that if* $n,m \in A'$ *and* $n \le m$ *then* $S(n) \supseteq S(m)$.

The proof of the following proposition is based on the proof of Theorem 4.1 of $[B_1]$.

Proposition 3.6. *The perfect image of a topological space having a well ranked base has a base which is the countable union of Noetherian collections.*

Proof. Suppose X is a topological space having a base $\mathcal{B} = \cup\{\mathcal{B}_n: 0 < n < \omega\}$ where for each $n \in \omega\backslash\{0\}$, $\mathcal{B}_n$ is Noetherian and has subinfinite rank and if $0 < m < n < \omega$ then $\mathcal{B}_m \subseteq \mathcal{B}_n$. Also, suppose f is a perfect mapping from X onto a topological space Y.

For $n,m \in \omega\backslash\{0\}$ let $F(n,m) = \{\mathcal{J} \subseteq \mathcal{B}_n: |\mathcal{J}| = m\}$, for $\mathcal{J} \in F(n,m)$ let $V(\mathcal{J}) = Y\backslash f(X\backslash\cup\mathcal{J})$ and let $\mathcal{U}(n,m) = \{V(\mathcal{J}): \mathcal{J} \in F(n,m)\}$. Let $\mathcal{B}' = \cup\{\mathcal{U}(n,m): n,m \in \omega\backslash\{0\}\}$. The collection $\mathcal{B}'$ is easily seen to be a base for Y. We will show that for each $n,m \in \omega\backslash\{0\}$ the collection $\mathcal{U}(n,m)$ is Noetherian.

Let $n,m \in \omega\backslash\{0\}$ and suppose $\mathcal{U}(n,m)$ is not Noetherian. Then for all $k \in \omega$ there is an $\mathcal{J}(k) \in F(n,m)$ such that if

$k < t < \omega$ then $V(\mathcal{J}(k)) \subset V(\mathcal{J}(t))$. Let $y(0) \in V(\mathcal{J}(0))$, $x(0) \in f^{-1}(y(0))$ and for all $k < \omega$, $A(k,0) \in \mathcal{J}(k)$ such that $x(0) \in A(k,0)$. Since $\{A(k,0): k < \omega\} \subseteq \mathcal{B}_n$ with nonempty intersection, by Lemma 3.5 there is an infinite set $S(1) \subseteq \omega$ such that if $k,t \in S(1)$ and $k < t$ then $A(k,0) \supseteq A(t,0)$.

Suppose for $0 < t < m$ the infinite set $S(t)$ and for all $k \in S(t)$ and all $j < t$, $A(k,j) \in \mathcal{J}(k)$ have been chosen satisfying the following conditions:

$(1)_t$ If $k \in S(t)$ and $i,j < t$ with $i \neq j$ then $A(k,i) \neq A(k,j)$.

$(2)_t$ If $i,j \in S(t)$ and $i < j$ then $\cup\{A(i,k): k < t\} \supseteq \cup\{A(j,k): k < t\}$.

Let $a(t) = \min S(t)$ and $b(t) = \min(S(t)\setminus\{a(t)\})$. Choose $y(t) \in V(\mathcal{J}(b(t)))$ such that $y(t) \notin V(\mathcal{J}(a(t)))$. Since $y(t) \notin V(\mathcal{J}(a(t)))$, we can choose an $x(t) \in f^{-1}(y(t))$ such that $x(t) \notin \cup\mathcal{J}(a(t))$. By $(2)_t$ for every $k \in S(t)\setminus\{a(t)\}$ we can choose an $A(k,t) \in \mathcal{J}(k)$ such that $x(t) \in A(k,t)$. The collection $\{A(k,t): k \in S(t)\setminus\{a(t)\}\} \subseteq \mathcal{B}_n$ has nonempty intersection. Thus by Lemma 3.5 there is an infinite $S(t+1) \subseteq S(t)\setminus\{a(t)\}$ such that if $i,j \in S(t+1)$ and $i < j$ then $A(i,t) \supseteq A(j,t)$. Conditions $(1)_{t+1}$ and $(2)_{t+1}$ hold.

Thus there is an infinite $S(m) \subseteq \omega$ and for each $k \in S(m)$ and $t < m$ a set $A(k,t) \in \mathcal{J}(k)$ satisfying $(1)_m$ and $(2)_m$. For each $k \in S(m)$, since $|\mathcal{J}(k)| = m$, by condition $(1)_m$, $\mathcal{J}(k) = \{A(k,i): i < m\}$. Let $k,j \in S(m)$ with $k < j$. By $(2)_m$, $\cup\mathcal{J}(k) \supseteq \cup\mathcal{J}(j)$ and so $V(\mathcal{J}(k)) \supseteq V(\mathcal{J}(j))$, a contradiction. Therefore $\mathcal{U}(n,m)$ is Noetherian.

References

[A] P. S. Alexandroff, *Some results in the theory of topological spaces obtained within the last twenty-five years*, Russian Math. Surveys 15 (1960), 23-83.

[Ar] A. V. Arhangl'skiĭ, *Mappings and spaces*, Russian Math. Surveys 21 (1966), 115-162.

[BL] H. R. Bennett and D. J. Lutzer, *A note on weak θ-refinability*, Gen. Top. Appl. 2 (1972), 49-54.

[B_1] D. K. Burke, *Paralindelof spaces and closed mappings*, Topology Proceedings 5 (1980), 47-58.

[B_2] ________, *Construction of spaces with a σ-minimal base*, Proc. Amer. Math. Soc. 81 (1981), 329-332.

[DRW] S. W. Davis, G. M. Reed and M. L. Wage, *Further results on weakly uniform bases*, Hous. J. Math. 2 (1976), 57-63.

[vD] E. K. van Douwen, *Noetherian bases* (manuscript).

[vDW] ________ and H. H. Wicke, *A real, weird topology on the reals*, Hous. J. Math. 3 (1977), 141-152.

[EL] R. Engelking and D. Lutzer, *Paracompactness in ordered spaces*, Fund. Math. 94 (1977), 49-58.

[Fl] W. G. Fleissner, *Applications of stationary sets in topology*, Surveys in General Topology, Academic Press, 1980, 163-193.

[F] O. Förster, *Spaces with bases of finite, point-finite or subinfinite rank*, thesis, University of Manitoba, Winnipeg, Manitoba, 1982.

[FG] ________ and G. Grabner, *The metacompactness of spaces with bases of subinfinite rank*, Top. Appl. 13 (1982), 115-121.

[Gr] G. Grabner, *Well partially ordered sets and local bases*, Topology Proceedings 6 (1981), 45-58.

[dG] J. de Groot, *Subcompactness and the Baire category theorem*, Indag. Math. 25 (1963), 761-767.

[G] G. Gruenhage, *Some results of spaces having an ortho-base or a base of subinfinite rank*, Topology Proceedings 2 (1977), 151-159.

[GN] ________ and P. Nyikos, *Spaces with bases of countable rank*, Gen. Top. Appl. 8 (1978), 233-257.

[HL] R. W. Heath and W. F. Lindgren, *Weakly uniform bases*, Hous. J. Math. 2 (1976), 85-90.

[Je] T. J. Jech, *Lectures in set theory with particular emphasis on the method of forcing*, Lecture Notes in Mathematics, No. 217, Springer-Verlag, New York, 1971.

[J] I. Juhasz, *Cardinal functions in topology*, Mathematical Centre, Amsterdam, 1975.

[LN] W. Lindgren and P. Nyikos, *Spaces with bases satisfying certain order and intersection properties*, Pac. J. Math. 66 (1976), 455-476.

[N_1] P. Nyikos, *Some surprising base properties in topology*, Studies in Topology, Academic Press, New York, 1975, 427-450.

[N_2] ________, *Some surprising base properties in topology II*, Set-Theoretic Topology, Academic Press, New York, 1977, 277-305.

[N_3] ________, *Countable small rank and cardinal invariants*, General Topology and its Relations to Modern Analysis and Algebra IV, Part B, Prague, 1976.

[N_4] ________, *Order-theoretic base axioms*, Surveys in General Topology, Academic Press, 1980, 367-398.

[Sc] B. M. Scott, *Local bases and product partial orders* (preprint).

[S] J. C. Smith, *A remark on irreducible spaces*, Proc. Amer. Math. Soc. 53 (1976), 133-139.

[SS] L. A. Steen and J. A. Seebach, *Counterexamples in topology*, Holt, Rinehart, and Winston, New York, 1970.

[WW] J. M. Worrell, Jr. and H. H. Wicke, *Characteristics of developable topological spaces*, Canad. J. Math. 17 (1965), 820-830.

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