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## PRIMITIVE $\sigma$ -SPACES

by

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## PRIMITIVE $\sigma$ -SPACES

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An interesting result due to Fletcher and Lindgren [FL<sub>2</sub>] is that spaces having primitive bases are  $\Theta$ -spaces. Wicke explored this relationship further by discussing the use of  $\Theta$ -space concepts in base of countable order theory [Wi<sub>1</sub>] and by giving a functional characterization of primitive base [Wi<sub>2</sub>]. His work led him to ask for a nontrivial property to add to  $\Theta$ -space in order to obtain primitive base.

In section 1 we provide an answer to Wicke's question by defining a property  $P$  and showing that a space has a primitive base if and only if it is a  $\Theta$ -space having property  $P$ . The author wishes to thank J. Chaber for pointing out that this property is not new and that it was defined and studied in [Ch<sub>2</sub>] as it relates to  $\sigma$ -spaces. Further ramifications (as it relates to spaces having a primitive base) including additional diagonal theorems are discussed here. We turn in section 2 to an exploration of the class of  $\sigma$ -scattered spaces as a special subclass of spaces having this property.

Unless otherwise stated, all spaces are regular  $T_0$  except in definitions, where only those separation properties stated are assumed.

### 1. Primitive $\sigma$ -Spaces

We begin with some definitions.

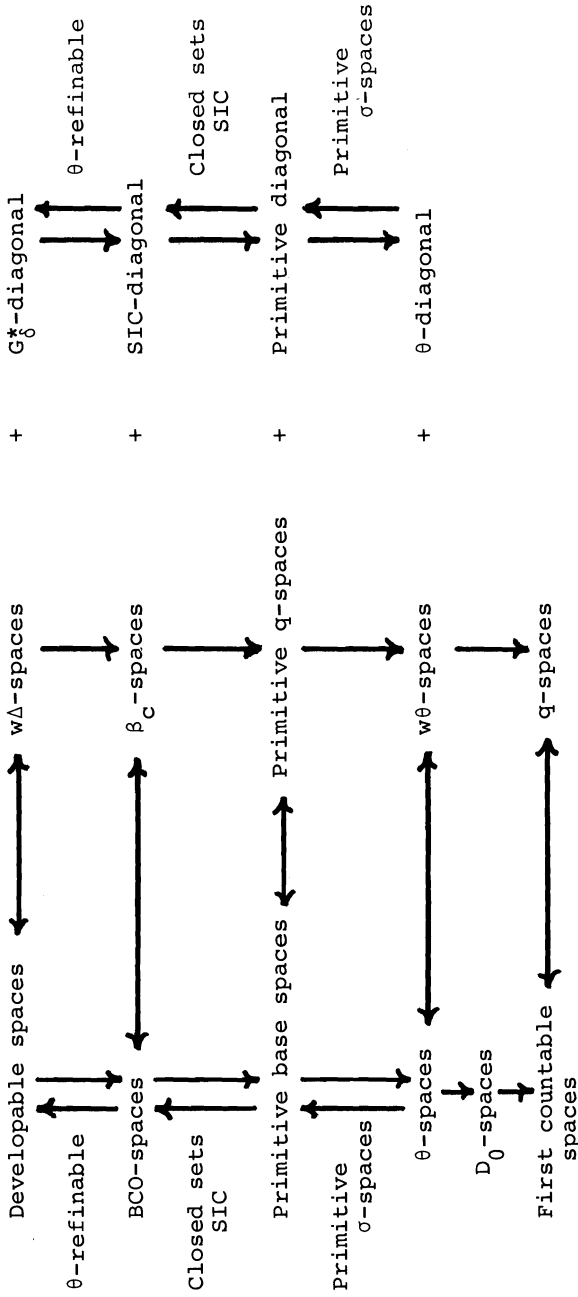


Diagram 1

1.1 *Definition* (Hodel, [Ho]). A space  $(X, \mathcal{J})$  is a  $\theta$ -space if there is a function  $g: \mathbb{N} \times X \rightarrow \mathcal{J}$  having the following properties.

a)  $x \in \bigcap_{n \in \mathbb{N}} g(n, x)$  for each  $x \in X$ .

b) If  $\{p, x_n\} \subseteq g(n, y_n)$  and  $y_n \in g(n, p)$  for each  $n \in \mathbb{N}$ , then  $p$  is a cluster point of  $\langle x_n: n \in \mathbb{N} \rangle$ . If in b) we just require that  $\langle x_n: n \in \mathbb{N} \rangle$  has a cluster point, we call  $X$  a  $w\theta$ -space. Fletcher and Lindgren [FL<sub>2</sub>] characterize  $\theta$ -spaces by replacing b) with the following.

b') If for each  $n \in \mathbb{N}$ ,  $y_n \in g(n, p)$  and  $p \in g(n, y_n)$ , then  $\{g(n, y_n): n \in \mathbb{N}\}$  is a base at  $p$ . If in b') we just require that  $\bigcap_{n \in \mathbb{N}} g(n, y_n) = \{p\}$ ,  $X$  is said to have a  $\theta$ -diagonal [Wi<sub>1</sub>].

The concept of primitive base was introduced by Worrell and Wicke in 1965. We state below characterizations of this concept and of the associated concepts of primitive  $q$ -space and primitive diagonal found in [Wi<sub>1</sub>, 2.3]. These may serve as definitions.

1.2 *Theorem* [Wi<sub>1</sub>, 2.3]. For each  $n \in \mathbb{N}$ , let  $\mathcal{W}_n$  be a well-ordered collection of open sets in  $X$  such that  $\mathcal{W}_n$  covers  $X$ . For each  $x \in X$ , let  $F(x, \mathcal{W}_n)$  denote the first element of  $\mathcal{W}_n$  that contains  $x$ .

a) If for each  $x \in X$ ,  $\{F(x, \mathcal{W}_n): n \in \mathbb{N}\}$  is a base at  $x$ , then  $X$  has a primitive base.

b) If  $y_n \in F(x, \mathcal{W}_n)$  for all  $n \in \mathbb{N}$  implies  $\langle y_n: n \in \mathbb{N} \rangle$  clusters, then  $X$  is a primitive  $q$ -space.

c) If for each  $x \in X$ ,  $\bigcap_{n \in \mathbb{N}} F(x, \mathcal{W}_n) = \{x\}$ , then  $X$  has a primitive diagonal.

From this characterization it is not hard to see that the class of all  $\theta$ -spaces contains the class of spaces having a primitive base. (For a more comprehensive view of some of the relationships existing among the spaces mentioned here, the reader is referred to Diagram 1 in the Introduction.) It is then a natural question to ask what nontrivial property a  $\theta$ -space must possess in order to have a primitive base. We answer this question after the following definition and observations.

**1.3 Definition.** Suppose a space  $X$  has a sequence  $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$  of well-ordered open covers.

a) If  $F(x, \mathcal{G}_n) = F(x_n, \mathcal{G}_n)$  for each  $n \in \mathbb{N}$  implies  $\langle x_n : n \in \mathbb{N} \rangle$  clusters to  $x$ , then  $X$  is called a *primitive  $\sigma$ -space* [Ch<sub>2</sub>].

b) If  $F(x, \mathcal{G}_n) = F(x_n, \mathcal{G}_n)$  for each  $n \in \mathbb{N}$  implies  $\langle x_n : n \in \mathbb{N} \rangle$  clusters, then  $X$  is called a *primitive  $w\sigma$ -space*.

c) If for each  $x \in X$ ,  $\bigcap_{n \in \mathbb{N}} \{y : F(x, \mathcal{G}_n) = F(y, \mathcal{G}_n)\} = \{x\}$ , then  $X$  is called a *primitive  $\sigma\#$ -space* [see Ch<sub>2</sub>].

**1.4 Questions.** a) Is every  $\sigma\#$ -space a primitive  $\sigma\#$ -space?

b) Is every  $w\sigma$ -space a primitive  $w\sigma$ -space?

Chaber poses a) in [Ch<sub>2</sub>]; we pose b) in view of the concept of  $w\sigma$ -space found in [FL<sub>1</sub>] and for the sake of completeness.

Clearly, a) Every  $T_1$  primitive  $\sigma$ -space is a primitive  $\sigma\#$ -space.

b) Every space having a primitive base is a primitive  $\sigma$ -space.

c) Every primitive  $q$ -space is a primitive  $w\sigma$ -space.

d) Every space with a primitive diagonal is a primitive  $\sigma\#$ -space.

*1.5 Theorem. A space  $X$  has a primitive base if and only if it is a  $\theta$ -space and a primitive  $\sigma$ -space.*

*Proof.* We need only show sufficiency. Assume  $X$  is well-ordered. Let  $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$  be a sequence witnessing the fact that  $X$  is a primitive  $\sigma$ -space, and let  $g$  be a function witnessing that  $X$  is a  $\theta$ -space. For each  $n \in \mathbb{N}$  and for each  $x \in X$ , let  $H(x, n) = F(x, \mathcal{G}_n) \cap g(n, x')$ , where  $x'$  is the first  $y$  in  $X$  such that  $F(y, \mathcal{G}_n) = F(x, \mathcal{G}_n)$  and  $x \in g(n, y)$ . Thus,  $\{H(x, n) : x \in X\}$  is a well-ordered cover of  $X$  such that for each  $x \in X$ , the first element containing  $x$  is  $H(x, n)$ .

Suppose  $H(x, n) = F(x, \mathcal{G}_n) \cap g(n, x_n)$  for each  $n \in \mathbb{N}$ . Then  $F(x, \mathcal{G}_n) = F(x_n, \mathcal{G}_n)$  for each  $n \in \mathbb{N}$ . Thus  $\langle x_n : n \in \mathbb{N} \rangle$  clusters to  $x$ . Since  $X$  is a  $\theta$ -space, it follows that  $\{H(x, n) : n \in \mathbb{N}\}$  is a base at  $x$ .

*1.6 Theorem. A primitive  $\sigma$ -space which is also a  $w\theta$ -space is a primitive  $q$ -space.*

*1.7 Theorem. A primitive  $\sigma$ -space with a  $\theta$ -diagonal has a primitive diagonal.*

The proofs of the above two theorems are obtained by making the appropriate changes in the proof of 1.5. The following examples show that we cannot improve upon Theorem 1.5 by using  $D_0$ -space or  $w\theta$ -space in place of  $\theta$ -space. (See Diagram 1 in the Introduction.)

1.8 *Examples.* a) Heath [He, 4.1] has an example of a bow-tie-type space which is a  $D_0$ -space (i.e., compact sets have countable character) but which is not a  $w\theta$ -space. This space is a primitive  $\sigma$ -space.

b) Let  $X = [0,1] \cup [2,3]$ . Define a topology on  $X$  as follows. If  $x \in [0,1]$ , then a neighborhood  $U'$  of  $x$  is a usual neighborhood  $U$  together with  $\{y + 2: y \in U \setminus \{x\}\}$ . If  $x \in [2,3]$ , then  $\{x\}$  is open. This space is a primitive  $\sigma$ -space and a  $w\theta$ -space, but not a  $\theta$ -space.

1.9 *Questions.* a) Is every primitive  $w\sigma$ -space which is also a  $\theta$ -space a primitive  $\sigma$ -space?

b) Does every primitive  $q$ -space with a  $\theta$ -diagonal have a primitive base?

If we try improving 1.5 by replacing primitive  $\sigma$ -space with primitive  $w\sigma$ -space, then a) is a natural question. This is related to b) since every primitive  $q$ -space is a primitive  $w\sigma$ -space and a  $w\theta$ -space, and every  $w\theta$ -space with a  $\theta$ -diagonal is a  $\theta$ -space. Thus, should our question a) be answered in the affirmative, so will b); should b) be answered in the negative, so will a). Question b) was posed by Wicke in [Wi<sub>1</sub>].

A strengthening of the concept of  $\theta$ -space along with the associated concept of  $w\theta$ -space was introduced by Fletcher and Lindgren in [FL<sub>1</sub>]. These concepts are called  $\underline{\theta}$ -space and  $\underline{w\theta}$ -space respectively. We add a definition to theirs and obtain results analogous to 1.5, 1.6 and 1.7 for primitive  $w\sigma$ -spaces. Hence, primitive  $\sigma$ -space can be

weakened if  $\theta$ -space is strengthened and still obtain primitive base.

1.10 *Definition* [FL<sub>1</sub>]. A space  $(X, \mathcal{J})$  is a  $\underline{\theta}$ -space provided there is a function  $g: \mathbb{N} \times X \rightarrow \mathcal{J}$  such that if for each  $n \in \mathbb{N}$ ,  $\{p, x_n\} \subseteq g(n, y_n)$  and  $\langle y_n: n \in \mathbb{N} \rangle$  has a cluster point, then  $p$  is a cluster point of  $\langle x_n: n \in \mathbb{N} \rangle$  (or equivalently, if  $p \in g(n, y_n)$  for  $n \in \mathbb{N}$  and  $\langle y_n: n \in \mathbb{N} \rangle$  has a cluster point, then  $\{g(n, y_n): n \in \mathbb{N}\}$  is a base at  $p$ ). If we just require that  $\langle x_n: n \in \mathbb{N} \rangle$  has a cluster point, we call  $X$  a  $w\underline{\theta}$ -space.

1.11 *Definition*. A space  $(X, \mathcal{J})$  is said to have a  $\underline{\theta}$ -diagonal if and only if there is a function  $g: \mathbb{N} \times X \rightarrow \mathcal{J}$  such that for each  $x \in X$ , if for some  $p \in X$  and each  $n \in \mathbb{N}$ ,  $x \in g(n, x_n)$  and  $x_n \in g(n, p)$ , then  $\bigcap_{n \in \mathbb{N}} g(n, x_n) = \{x\}$ .

Clearly,  $\underline{\theta}$ -spaces have  $\underline{\theta}$ -diagonals and spaces with  $\underline{\theta}$ -diagonals have  $\theta$ -diagonals.

1.12 *Theorem*. A primitive  $w\theta$ -space which is also a  $\underline{\theta}$ -space has a primitive base.

1.13 *Theorem*. A primitive  $w\theta$ -space which is also a  $w\underline{\theta}$ -space is a primitive  $q$ -space.

1.14 *Theorem*. A primitive  $w\theta$ -space with a  $\underline{\theta}$ -diagonal has a primitive diagonal.

The proofs of the above are all analogous to the proof of 1.5.



1.15 *Examples.* a)  $\omega_1$ , the set of all ordinals less than the first uncountable ordinal, with the order topology has a base of countable order (hence, a primitive base) but is not a  $\theta$ -space. Thus, the converse of Theorem 1.12 is false (cf. 1.5).

b) The Michael line (i.e., the set of real numbers  $\mathbb{R}$  with the topology generated by the usual topology for the reals together with the singletons of the irrational points) shows that a primitive  $\sigma$ -space which is also a  $\theta$ -space need not have a base of countable order.

Clearly, if we demand of primitive  $w\sigma$  (resp.  $w\theta$ )-spaces that  $\langle x_n : n \in \mathbb{N} \rangle$  clusters to  $y \in \bigcap_{n \in \mathbb{N}} F(x_n, \mathcal{G}_n)$  ( $\bigcap_{n \in \mathbb{N}} G(n, y_n)$ ), then we obtain results such as the following.

1.16 *Theorem* [FL<sub>2</sub>, Corollary to 2.2]. *A  $w\theta$ -space with a  $\theta$ -diagonal is a  $\theta$ -space.*

This raises the following questions.

1.17 *Question.* Does every regular  $w\theta$  (primitive  $w\sigma$ ,  $w\theta$ )-space have a  $w\theta$  (primitive  $w\sigma$ ,  $w\theta$ )-function which satisfies the stronger condition above?

This type of question (cf. 1.20 below) arises again concerning the concept of monotonic  $\beta$ -space in view of Theorem 1.19 and the remark which follows it. This theorem illustrates the "completeness" required of  $\theta$ -spaces in order to obtain spaces having a base of countable order.

1.18 *Definition* (Chaber [Ch<sub>1</sub>, 1.6]). A space  $X$  is called a *monotonic  $\beta$ -space* if and only if for all  $x \in X$  there is a decreasing sequence  $\langle \beta_n(x) : n \in \mathbb{N} \rangle$  of bases at  $x$  such that if, for all  $n \in \mathbb{N}$ ,  $B_n \in \beta_n(x_n)$ ,  $B_{n+1} \subseteq B_n$ , and  $\bigcap_{n \in \mathbb{N}} B_n \neq \emptyset$ , then  $\langle x_n : n \in \mathbb{N} \rangle$  has a cluster point.

1.19 *Theorem* [Wi<sub>1</sub>, 5.1]. Let  $X$  be a regular  $T_1$  monotonic  $\beta$ -space. Then the following are equivalent:

- a)  $X$  has a base of countable order.
- b)  $X$  is a  $\theta$ -space.

If in the definition of monotonic  $\beta$ -space one required the stronger condition that  $\langle x_n : n \in \mathbb{N} \rangle$  has a cluster point  $y \in \bigcap_{n \in \mathbb{N}} B_n$ , then it is easier to argue than the above that a  $\theta$ -space which is this stronger monotonic  $\beta$ -space has a base of countable order. Thus, we ask the following.

1.20 *Question*. Does every regular monotonic  $\beta$ -space satisfy the stronger condition above?

## 2. $\sigma$ -Scattered Spaces

Spaces which are  $\sigma$ -scattered (including  $\sigma$ -(closed and scattered) and scattered spaces) are special cases of primitive  $\sigma$ -spaces. This might be expected in light of the following which are found all together in [WW].

- a) Every  $T_1$  first countable scattered space has a monotonically complete base of countable order.
- b) Every  $\sigma$ -(closed and scattered)  $T_1$  first countable space has a base of countable order.
- c) Every  $\sigma$ -scattered first countable space has a primitive base.

It is clear then that the property of being  $\sigma$ -scattered has a rather strong aspect of primitivity. This must, in fact, be stronger than primitive  $\sigma$ -space since we have c) above in contrast to Theorem 1.5 where  $\theta$ -space appears in place of first countable.

One of our interests here is to provide functional characterizations of these various scattered concepts in view of our characterization of primitive  $\sigma$ -space and later in view of the functional characterization of primitive base of  $[Wi_2]$ . Using well-ordering we may in fact formulate these concepts by means of the following.

**2.1 Definition.** Let  $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$  be a sequence of well-ordered open covers of a space  $X$ . For each  $x \in X$  and  $n \in \mathbb{N}$ , let  $F(x, n)$  be the first element of  $\mathcal{G}_n$  that contains  $x$ .

a)  $F$  is called *s-primitive*  $\sigma$  if and only if for each  $x \in X$ , there is  $n_x \in \mathbb{N}$  such that  $\{y : F(y, m) = F(x, m)\} = \{x\}$  for all  $m \geq n_x$ .

b)  $F$  is called *rank 1 s-primitive*  $\sigma$  if and only if there is  $n \in \mathbb{N}$  such that for each  $x \in X$ ,  $\{y : F(y, m) = F(x, m)\} = \{x\}$  for all  $m \geq n$ .

c)  $F$  is called *closed s-primitive*  $\sigma$  if and only if  $F$  is s-primitive  $\sigma$  and for each  $n \in \mathbb{N}$ ,  $\{x : \{y : F(y, n) = F(x, n)\} = \{x\}\}$  is closed.

**2.2 Definition.** A space  $X$  is *scattered* if and only if every subset  $A \subseteq X$  has an isolated point (i.e., there is an  $a \in A$  such that  $\{a\}$  is open in  $A$ ). It is  *$\sigma$ -scattered* if it is the union of a countable collection of scattered

subspaces; if each of these scattered subsets is closed, the space is called  $\sigma$ -(closed and scattered).

**2.3 Theorem.** *A space  $X$  is scattered if and only if it has a sequence  $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$  of well-ordered open covers such that  $F$  is rank 1  $s$ -primitive  $\sigma$ .*

*Proof.* Suppose  $X$  is scattered and is well-ordered. Let  $x_0$  be the first isolated point of  $X$ . There is an open set  $U_0$  such that  $U_0 \cap X = \{x_0\}$ . Having chosen  $x_\alpha$  and  $U_\alpha$  for all  $\alpha < \beta$ , let  $x_\beta$  be the first isolated point of  $X \setminus \{x_\alpha : \alpha < \beta\}$  provided this is nonempty. Then there is an open set  $U_\beta$  such that  $U_\beta \cap X \setminus \{x_\alpha : \alpha < \beta\} = \{x_\beta\}$ . Eventually  $X \setminus \{x_\alpha : \alpha < \beta\} = \emptyset$ ; thus we have covered  $X$ .

Now we have a well-ordered open cover  $\mathcal{U}$  of  $X$  such that for each  $x \in X$ ,  $\{y : F(y, \mathcal{U}) = F(x, \mathcal{U})\} = \{x\}$ . Defining  $\mathcal{G}_n = \mathcal{U}$  for all  $n \in \mathbb{N}$  gives us a rank 1  $s$ -primitive  $\sigma$  function with respect to  $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ .

Suppose  $X$  has a rank 1  $s$ -primitive  $\sigma$  function with respect to some sequence  $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$  of well-ordered open covers and that  $n \in \mathbb{N}$  witnesses that it is rank 1. Let  $A \subseteq X$  be nonempty,  $G \in \mathcal{G}_n$  be the first element of  $\mathcal{G}_n$  which meets  $A$ , and  $x \in G \cap A$ . Then  $F(x, n) = G$ . If  $y \in G \cap A$ , then also  $F(y, n) = G$ . Thus  $x = y$ . Hence,  $x$  is an isolated point of  $A$ .

The proofs of the following two theorems are similar to the proof of 2.3 and are left to the reader.

**2.4 Theorem.** *A space  $X$  is  $\sigma$ -(closed and scattered) if and only if it has a sequence  $\langle \mathcal{G}_n: n \in \mathbb{N} \rangle$  of well-ordered open covers such that  $F$  is closed  $s$ -primitive  $\sigma$ .*

**2.5 Theorem.** *A space  $X$  is  $\sigma$ -scattered if and only if it has a sequence  $\langle \mathcal{G}_n: n \in \mathbb{N} \rangle$  of well-ordered open covers such that  $F$  is  $s$ -primitive  $\sigma$ .*

From the above characterizations, we can now see the precise relationship of primitive  $\sigma$ -spaces to those which are  $\sigma$ -scattered. These characterizations, however, are extrinsic in that the function  $F$  is defined by a sequence of well-orderings. In order to eliminate the explicit well-orderings we provide purely functional characterizations by applying Wicke's concept of an initial function found in [Wi<sub>2</sub>].

**2.6 Definition** (Wicke, [Wi<sub>2</sub>]). A function  $h: \mathbb{N} \times X \rightarrow \mathcal{J}$  is called *initial* if and only if for  $j \in \mathbb{N}$  and  $A \subseteq X$ , if  $A \neq \emptyset$ , there exists  $a \in A$  such that for all  $b \in A$ , either  $h(j,a) = h(j,b)$  or  $b \notin h(j,a)$ .

This gives rise to the following characterization of primitive base.

**2.7 Theorem** [Wi<sub>2</sub>, 2.2]. *A space has a primitive base if and only if it has an initial function which is first countable (i.e., for all  $p \in X$ ,  $\{h(j,p): j \in \mathbb{N}\}$  is a base at  $p$ ).*

In the same manner, we obtain characterizations of scattered,  $\sigma$ -(closed and scattered),  $\sigma$ -scattered and primitive  $\sigma$ . We state here only the characterization of primitive  $\sigma$  as the others will be apparent in light of Definition 2.1.

**2.8 Theorem.** *A space  $X$  is a primitive  $\sigma$ -space if and only if it has an initial function  $h$  such that  $h(n, x) = h(n, x_n)$  for all  $n \in \mathbb{N}$  implies  $\langle x_n : n \in \mathbb{N} \rangle$  clusters to  $x$ .*

*Proof.* Suppose  $X$  is a primitive  $\sigma$ -space, and let  $\langle \mathcal{U}_n : n \in \mathbb{N} \rangle$  witness this. It is not hard to show that the function  $h$  defined by  $h(n, x) = F(x, \mathcal{U}_n)$  is an initial function satisfying the condition above.

To show sufficiency, suppose  $X$  has an initial function satisfying the condition above. Following the proof of [Wi<sub>2</sub>, Theorem 2.2], we obtain a sequence  $\langle \mathcal{W}_j : j \in \mathbb{N} \rangle$  of well-ordered open covers such that  $F(x, \mathcal{W}_j) = h(j, x)$ .

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