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by

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1. Introduction

All spaces under discussion are separable metric.

The Anderson-Kadec Theorem (see [BP] for background information) that every infinite-dimensional Fréchet space is homeomorphic to the Hilbert space ℓ_2, suggests the question whether "nice" subsets of Fréchet spaces are always homeomorphic to "nice" subsets of ℓ_2. As far as I know it is open whether every locally convex real vector space is homeomorphic to a linear subspace of ℓ_2. Let us consider $\mathbb{R}^\infty$ to be a vector space over the rationals Q. In this note we will show that there is a linear subspace L of $\mathbb{R}^\infty$ that is not homeomorphic to a normed vector space over Q.

2. Preliminaries

A (topological) vector space over Q is a topological space X that is a vector space over Q such that the algebraic operations

$$\langle x,y \rangle \to x + y, \quad \text{and}$$

$$x \to qx \quad (q \in Q \text{ fixed})$$

are continuous. A vector space over Q will be called a *rational vector space* from now on. As usual, a rational vector space L is called *normed* if there is a function $||\cdot|| : L \to \mathbb{R}^+$ such that

$$\|x + y\| \leq \|x\| + \|y\|,$$
$$\|qx\| = |q| \cdot \|x\|,$$
$$\|x\| = 0 \leftrightarrow x = 0$$

for all $x,y \in L$, $q \in \mathbb{Q}$, while moreover the metric

$$d(x,y) = \|x - y\|$$

generates the topology on L. Observe that $\|x\| \in \mathbb{R}^+$ for every $x \in L$ and not, as some might expect, that $\|x\|$ is always rational.

3. The Construction

In this section we will construct the example that was announced in the introduction.

3.1 *Theorem. Let* X *be a topologically complete vector space over* $\mathbb{R}$ *of dimension at least* 2. *Then* X *contains a connected subspace* L *such that*

(1) *if* $x,y \in L$ *and* $s,t \in \mathbb{Q}$ *then* $sx + ty \in L$,

(2) *if* $h: L \to L$ *is any autohomeomorphism then there are* $q \in \mathbb{Q}\setminus\{0\}$ *and* $y \in L$ *such that* $h(x) = qx + y$, *for every* $x \in L$,

(3) L *intersects every Cantor set in* X.

The proof of this result, except for trivial modifications, is the same as the proof of [vM, Theorem 3.1] and will therefore be omitted.

Now let $X = \mathbb{R}^\infty$ and let $L \subseteq X$ be as in Theorem 3.1. By (1), L is a rational vector space and we claim that L is as required. Striving for a contradiction, let M be a normed rational vector space and let $h: L \to M$ be a homeomorphism. By $\underline{0}$ we will denote the point $(0,0,\cdots) \in \mathbb{R}^\infty$.

Since M is homogeneous, without loss of generality we may assume that $h(\underline{0}) = 0$. To avoid confusion, the algebraic operations on M will be denoted by $\oplus$ and $\cdot$, respectively. Define $\gamma: M \to M$ by $\gamma(x) = x \oplus x$. Then γ is a homeomorphism of M which implies that $\xi = h^{-1}\gamma h$ is a homeomorphism of L. Observe that

$$\xi(\underline{0}) = h^{-1}\gamma h(\underline{0}) = h^{-1}\gamma(0) = h^{-1}(0) = \underline{0}.$$

By (2) there exist $q \in Q$ and $y \in L$ such that

$$\xi(x) = qx + y$$

for every $x \in L$. Since $\xi(\underline{0}) = \underline{0}$ it follows that $y = \underline{0}$, whence $\xi(x) = qx$ for every $x \in L$. Let $U = \{x \in M: \|x\| < 1\}$. Then U is an open neighborhood of 0 in M, whence $h^{-1}(U)$ is an open neighborhood of $\underline{0}$ in L. Choose an open neighborhood V of 0 in $\mathbb{R}$ and an $n \in \mathbb{N}$ such that

$$W = (\underbrace{V \times V \times V \times \cdots \times V}_{n\times} \times \mathbb{R} \times \mathbb{R} \times \cdots) \cap L \subseteq h^{-1}(U).$$

By (3), there is a point $x \in W\setminus\{\underline{0}\}$ such that $x_i = 0$ for every $i \leq n$. Then $Qx \subseteq W$ and from this we conclude that $\{\xi^n(x): n \in \mathbb{N}\} \subseteq W$. Put $y = h(x)$. Then

$$\xi^n(h^{-1}(y)) = (h^{-1}\gamma h)^n(h^{-1}(y)) = h^{-1}\gamma^n hh^{-1}(y)$$
$$= h^{-1}\gamma^n(y) \in W \subseteq h^{-1}(U)$$

for every $n \in \mathbb{N}$. Consequently, $\gamma^n(y) \in U$, $n \in \mathbb{N}$. Let $\varepsilon = \|y\|$. Observe that $\varepsilon > 0$. Take $n \in \mathbb{N}$ so large that $2^n\varepsilon > 1$. Then $\|\gamma^n(y)\| = \|2^n \cdot y\| = 2^n\|y\| = 2^n\varepsilon > 1$, which is a contradiction.

References

[BP] C. Bessaga and A. Pelczyński, *Selected topics in infinite-dimensional topology*, PWN, Warsaw, 1975.

[vM] J. van Mill, *A topological group having no homeomorphisms other than translations*, Trans. Amer. Math. Soc. 280 (1983), 491-498.

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