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Abstract. We prove that if $\mathcal{F}$ is a maximal closed filter on a space X then $|\mathcal{F}| = o(X)$ (= the number of closed subsets of X), and there exists a space X and a prime closed filter $\mathcal{F}$ on X with $|\mathcal{F}| < o(X)$. Further, cardinals of the form $|\mathcal{F}|$, where $\mathcal{F}$ is a prime closed filter on a T_2-space, are never singular, strong limit cardinals. We also consider the related notion of an astral closed filter. These results about closed filters are used in the study of some weak covering properties. A. V. Arhangel'skii has introduced the three covering properties ultrapure, astral and pure which are all weaker than weak $\delta\Theta$-refinable. He proved that if $\mathcal{P}$ is any one of these three properties then every countably compact space having property $\mathcal{P}$ is compact. J. M. Worrell, Jr., and H. H. Wicke proved that if Q^r is the property defined by restricting the definition of weak $\delta\Theta$-refinable to apply only to open covers of regular cardinality, then every countably compact space having property Q^r is compact. In this paper we answer the question: If $\mathcal{P}$ is one of the three properties of Arhangel'skii, then is every countably compact space having property $\mathcal{P}^r$ compact? The answer is "yes" for $\mathcal{P}$ = ultrapure, and is independent of and consistent with ZFC for $\mathcal{P}$ = astral or pure.

1. Introduction and Basic Results

We recall several standard definitions (see [W]).

Definition. A family of closed subsets $\mathcal{F}$ of a space X is called a (free) *closed filter* provided the following hold:

1.0. $\mathcal{F} \neq \emptyset$,

1.1. $\emptyset \notin \mathcal{F}$,

1.2. $\cap\mathcal{F} = \emptyset$,

1.3. If $F \in \mathcal{F}$ and $H \in \mathcal{F}$ then $F \cap H \in \mathcal{F}$,

1.4. If $F \in \mathcal{F}$ and $F \subset H$ and H is closed, then $H \in \mathcal{F}$.

Because of our interest in open covers, we assume 1.2 throughout this paper and drop the adjective "free." A closed filter $\mathcal{F}$ is called a *maximal closed filter* provided $\mathcal{F}$ is not properly contained in any closed filter. This is equivalent to saying that $\mathcal{F}$ satisfies,

1.5. For every closed set H, either $H \in \mathcal{F}$ or there is an $F \in \mathcal{F}$ such that $F \cap H = \emptyset$.

A closed filter is called *prime* provided it satisfies,

1.6. If F, G are closed and $F \cup G \in \mathcal{F}$, then $F \in \mathcal{F}$ or $G \in \mathcal{F}$.

We also consider the following concept which was introduced by Arhangel'skii in order to define astral spaces. A closed filter is called *astral* provided it satisfies the following 1.7, 1.8 :

1.7. If F_i are closed sets for $i < \omega$, and $\cup\{F_i: i < \omega\} \in \mathcal{F}$, then there exists $i < \omega$ such that $F_i \in \mathcal{F}$.

1.8. If $F_i \in \mathcal{F}$ for all $i < \omega$, then $\cap\{F_i: i < \omega\} \in \mathcal{F}$.

In the remainder of this section, and in §2 and §3 we are concerned with the cardinalities of maximal, astral and prime closed filters. We apply some of these results in §4 to prove the topological results mentioned in the abstract. Since we are only interested in closed filters in this paper we may drop the adjective "closed." When we speak of an ultrafilter, we mean an ultrafilter of sets.

1.9. *Proposition. If $\mathcal{F}$ is a family of closed subsets of a T_1-space X satisfying 1.2 and 1.4, then $|X| \leq |\mathcal{F}|$.*

Proof. Assume false, i.e., assume that there is a space X and a family $\mathcal{F}$ of closed subsets of X satisfying 1.2 and 1.4 such that $|X| > |\mathcal{F}|$. Pick such a space X having smallest possible cardinality. Clearly for each $F \in \mathcal{F}$ we have $|X - F| < |X|$, or else by the T_1 property and 1.4 $\{F \cup \{x\}: x \in X - F\} \subset \mathcal{F}$. By 1.2 $X = \cup\{X - F: F \in \mathcal{F}\}$; so $|X| \leq \Sigma\{|X - F|: F \in \mathcal{F}\}$. Thus $|\mathcal{F}| \geq cf(|X|)$, hence $|X| \neq cf(|X|)$ by our assumption; so we see that $|X|$ is singular. Let $|X| = \lambda = \Sigma\{\lambda_\xi: \mu < k\}$ where $k = cf(\lambda)$ and $\mu < \xi < k$ imply $\lambda_\mu < \lambda_\xi < \lambda$. Well-order X as $X = \{x_\alpha: \alpha < \lambda\}$

and put $X_\mu = \{x_\alpha : \alpha < \lambda_\mu\}$ for all $\mu < k$. By our choice of X it is easy to see that for every $\mu < k$, we may choose a $F_\mu \in \mathcal{F}$ such that $F_\mu \cap X_\mu = \emptyset$. Then $\{F_\mu \cup \{x\} : x \in X - F_\mu\} \subset \mathcal{F}$ for all $\mu < k$. This implies that $\lambda \leq |\mathcal{F}|$, a contradiction. This completes the proof.

Recall that for a space X, $o(X)$ denotes the number of open subsets of X. If $\mathcal{F}$ is one of the three filters we are considering on a T_1-space X, then

1.10. $|X| \leq |\mathcal{F}| \leq o(X) \leq 2^{|X|}$.

1.11. *Proposition. If $\mathcal{F}$ is a family of closed sets in X satisfying 1.4, and 1.5, then $|\mathcal{F}| = o(X)$.*

Proof. For each F in $\mathcal{F}$, let $D(F) = \{H : H$ is closed in X and $F \cap H = \emptyset\}$. Since the map $H \mapsto F \cup H$ from $D(F)$ into $\mathcal{F}$ is one-to-one, we see that $|D(F)| \leq |\mathcal{F}|$. By 1.5, we have

$$o(X) = |\{H : H \text{ is closed in } X\}| \leq |\mathcal{F}| + \Sigma\{|D(F)| : F \in \mathcal{F}\} \leq |\mathcal{F}|.$$

1.12. *Corollary. If $\mathcal{F}$ is a maximal closed filter on a space X then $|\mathcal{F}| = o(X)$.*

2. Cardinalities of Prime Filters of Closed Sets

In contrast to Corollary 1.12 for maximal filters, we have

2.1. *Example.* There is a space $X \subset \beta(\omega)$ and a prime filter of closed sets $\mathcal{F}$ on X with $|\mathcal{F}| < o(X)$.

This example is based on the following two results.

2.2. *Lemma. If* k *is the first cardinal such that* $2^k > 2^\omega$, *then* k *is regular and* $\omega_1 \leq k \leq 2^\omega$.

The proof of this lemma is a simple exercise or one may use Theorem 18 in [J].

Recall that a space X is called a P-*space* provided that every countable intersection of open sets in X is open. Let ω^* denote $\beta(\omega) - \omega$ (the Stone-Čech remainder of the natural numbers ω).

2.3. *Theorem* (see [vM], 4.4.4). *If* X *is a* P-*space of weight* $\leq 2^\omega$ *then* X *can be embedded into* ω^*.

Proof of 2.1. Let k be the cardinal mentioned in Lemma 2.2, and let $Y = k + 1$ with the smallest topology containing the order topology in which every $\alpha < k$ is isolated. Clearly Y is a P-space of weight k (having only the last point k not isolated); so by Theorem 2.3, there is an embedding $h: Y \to \omega^*$. The space we seek is

$$X = \omega \cup \{h(\alpha): \alpha < k\} \subset \beta(\omega).$$

We define a prime filter $\mathcal{F}$ on X as follows. Let $u = h(k)$, and regarding u as an ultrafilter on ω, put

$$\mathcal{F} = \{H \subset X: H \text{ is closed and } H \cap \omega \in u\}.$$

That $\mathcal{F}$ is prime follows from the facts that $u \notin X$ (so $\cap\mathcal{F} = \emptyset$) and that $\{cl_{\beta\omega}(U) \cap X: U \in u\}$ is a base for $\mathcal{F}$ (so 1.6 holds). We now calculate $|\mathcal{F}|$. For each $U \in u$, let $P(U) = \{F \in \mathcal{F}: U \subset F\}$. Then $\mathcal{F} = \cup\{P(U); U \in u\}$. We show that for each U, we have $|P(U)| \leq 2^\omega$. This follows because $cl_{\beta\omega}(U)$ is a neighborhood of u in $\beta\omega$ and h is an embedding.

Thus $|X - cl_{\beta\omega}(U)| < k$, hence by the definition of k

$$(*) \quad 2^{|X - cl_{\beta\omega}(U)|} \leq 2^{\omega}.$$

It follows that if $U \subset H$ and H is closed, then $(X - H)$ is open in the subspace $(X - cl_{\beta\omega}(U))$ of X. Since the map $H \mapsto (X - H)$ from $P(U)$ into the subspace topology on $(X - cl_{\beta\omega}(U))$ is one-to-one, we see by (*) that $|P(U)| \leq 2^{\omega}$. This shows that $|\mathcal{F}| \leq 2^{\omega}$. Since the space X has a discrete subspace of cardinality k, we have $o(X) \geq 2^k$. This gives $|X| = k \leq |\mathcal{F}| = 2^{\omega} < 2^k = o(X)$.

It is consistent that every cardinal number of the form $|o(X)|$, where X is a T_2-space, is regular and also that $o(X) = 2^k$ for some k; see [J_2, Chapter 4]. Thus by Proposition 1.12, the same can be said for cardinals of the form $|\mathcal{F}|$ where $\mathcal{F}$ is a maximal closed filter on a T_2-space. We now show that, despite Example 2.1, these same consistency results hold for cardinals of the form $|\mathcal{P}|$ where $\mathcal{P}$ is a prime closed filter on a T_2-space. We use the following theorem of A. Hajnal and I. Juhász ([HJ] or [J_2, 4.5]) which is the basic result used to prove the results about $o(X)$.

2.4. *Theorem* (Hajnal and Juhász). *If* X *is a* T_2*-space, then* $o(X)$ *is not a singular, strong limit cardinal.*

2.5. *Theorem. If* $\mathcal{P}$ *is a prime closed filter on a* T_2*-space* X*, then* $|\mathcal{P}|$ *is not a singular, strong limit cardinal.*

Proof. Assume false, i.e., assume that there is a T_2-space X and a prime filter $\mathcal{P}$ on X with $|\mathcal{P}| = \lambda$, where

λ is a singular, strong limit cardinal. CLAIM: For every P in $\mathcal{P}$ we have $|X - P| < \lambda$. If the claim is not true, then there exists $P \in \mathcal{P}$ such that $|X - P| = \lambda$; so by Theorem 2.4, the subspace $(X - P)$ of X has at least λ^+ relatively closed sets. For each relatively closed set $F \subset (X - P)$ we assign the closed set $(\mathrm{cl}_X(F) \cup P) \in \mathcal{P}$. This assignment, however, is one-to-one, which implies that $|\mathcal{P}| \geq \lambda^+$. This contradiction establishes the claim. We now use this claim to get a contradiction. Let $k = \mathrm{cf}(\lambda)$ and let $\{\lambda_\alpha: \alpha < k\}$ be a set of cardinals less than λ such that $\lambda = \Sigma\{\lambda_\alpha: \alpha < k\}$. By transfinite induction on k we construct a family $\{U_\alpha: \alpha < k\}$ of mutually disjoint open subsets of X such that $\lambda_\alpha \leq |U_\alpha| < \lambda$ for all $\alpha < k$. Suppose we have constructed U_α for $\alpha < \gamma$ where $\gamma < k$. Construct U_γ as follows. Let $Y = \cup\{U_\alpha: \alpha < \gamma\}$. Then $|Y| \leq \Sigma\{|U_\alpha|: \alpha < \gamma\} < \lambda$. By [$J_1$, 2.4], $|\overline{Y}| \leq \exp\exp|Y| < \lambda$; so $(X - \overline{Y})$ is an open subset of X with $|X - \overline{Y}| = \lambda$. For $\beta < k$, let

$$B_\beta = \{x \in X - \overline{Y}\text{: there is a neighborhood } U \text{ of } x \text{ with } |U| \leq \lambda_\beta\}$$

By the claim and (1.2), every point in X has such a neighborhood, hence $X - \overline{Y} = \cup\{B_\beta: \beta < k\}$. Thus, there exists $\beta < k$ such that $|B_\beta| \geq \lambda_\gamma$. Let B be a subset of B_β with $|B| = \lambda_\gamma$. For each b in B, let $U_b \subset X - \overline{Y}$ be an open neighborhood of b with $|U_b| \leq \lambda_\beta$. Put $U_\gamma = \cup\{U_b: b \in B\}$. This completes the induction. Now let S and L be two disjoint subsets of k such that $S \cup L = k$, and $|S| = |L| = k$. Put $U_S = \cup\{U_\alpha: \alpha \in S\}$, and $U_L = \cup\{U_\alpha: \alpha \in L\}$. Since $X = \overline{U}_S \cup \overline{U}_L \cup (X - (U_S \cup U_L))$, and $\mathcal{P}$ is prime, one of these three closed sets must be in $\mathcal{P}$, but this contradicts the claim.

Using Theorem 2.5, and following the proofs of the analogous results about o(X) in [J_2, Chapter 4] we get

2.6. *Corollary. Let $\mathcal{P}$ be a prime closed filter on a* T_2*-space* X. *Then* (a) GCH *implies that* $|\mathcal{P}|$ *is regular, and* (b) GCH *and there exist no inaccessible cardinals imply that* $|\mathcal{P}| = 2^k$ *for some* k.

3. Astral Filters and Measurable Cardinals

Every countably compact, non-compact space has (by Zorn's Lemma) a maximal closed filter, and it is easy to see that on a countably compact space, every maximal filter is astral. Thus astral filters exist. On the other hand, we have the next example.

3.1. *Example. There exists a non-compact space* $X \subset \beta(\omega)$ *such that* X *has no astral filters.*

Proof. We may take for X the space of Example 2.1. Suppose that $\mathcal{A}$ is an astral filter on X. By 1.2 and 1.8, there exists A in $\mathcal{A}$ such that A misses the countable set ω. Thus, $(X - \omega) \in \mathcal{A}$. Since $(X - \omega)$ is a closed discrete subset of X, we have that $u = \{A \cap (X - \omega) : A \in \mathcal{A}\}$ is a non-principal, ultrafilter on $(X - \omega)$ which is ω^+-complete (i.e., closed under countable intersections). But this is impossible since $|X - \omega| = k \leq 2^\omega$ is non-measurable [GJ, 12.5].

We ask the question: Does there exist a space X and an astral filter $\mathcal{A}$ on X with $|\mathcal{A}| < \dot{o}(X)$? It is natural to try to generalize the embedding construction of Example 2.1. As in 3.1, however, such a procedure might imply the existence of an uncountable measurable cardinal. In fact, we do not know if it is possible to construct such a pair X, $\mathcal{A}$ in ZFC, but we can construct such a pair by using a measurable cardinal k. It is not clear how to apply (the general version of) the embedding theorem 2.3 since we need h(k) to be k-complete, but it is easy enough to construct the subspace of $\beta(k)$ directly using the following result.

3.3. *Lemma. If there exists a measurable cardinal* $k > \omega$ *and* $2^k = k^+$ *then there exists a* k-*complete, uniform ultrafilter* p *on* k *having a base* $\{B_\alpha: \alpha < k^+\}$ *such that for all* $\alpha < \beta < k^+$ *we have* $|B_\beta - B_\alpha| < k$ *and* $|B_\alpha - B_\beta| = k$.

Proof. We do not know a specific reference for this result, but it follows easily from [CN, 8.26, and 9.6(d)] which gives a k-complete uniform ultrafilter p on k such that for every partition $\{d_\xi: \xi < k\}$ on k there exists P in p such that $|\{\xi < k: |P \cap d_\xi| > 1\}| \leq 1$. This p can be seen to have the required base in a manner similar to Walter Rudin's construction of a P-point in ω^* (the point p will be a P_{k^+}-point in U(k), the set of uniform ultrafilters on k).

3.4. *Example. If there exists a measurable cardinal* $k > \omega$ *and* $2^k = k^+$, *then there exists a space* $X \subset \beta(k)$ *and an astral filter* $\mathcal{A}$ *on* X *such that* $|\mathcal{A}| < o(X)$.

Proof. Let p be the ultrafilter in Lemma 3.3, and pick points

$$x_\alpha \in cl_{\beta k}(B_\alpha - B_{\alpha+1}) \cap U(k)$$

for all $\alpha < k^+$. Then $\{x_\alpha : \alpha < k^+\}$ is a discrete subset of $U(k)$. Take $X = k \cup \{x_\alpha : \alpha < k\}$ and $\mathcal{A} = \{H \subset X : H$ is closed in X and $H \cap k \in p\}$.

4. Ultrapure, Astral and Pure Spaces

4.1. *Definition* (Arhangel'skii [A]). A space X is called *ultrapure* (resp. *astral*, or *pure*) provided every family of closed sets (resp. astral closed filter, or maximal closed filter) has a δ-suspension.

In this paper we do not need to know exactly what a δ-suspension is because we only have to be concerned with the cardinalities of closed filters.

4.2. *Theorem* (Arhangel'skii [A]).

(a) *Every weakly $\delta\Theta$-refinable space is an ultrapure space.*

(b) *ultrapure → astral → pure.*

(c) *Every countably compact, pure space is compact.*

4.3. *Definition.* If $\mathcal{P}$ is a property of the form "for every family $\mathcal{F}$ with property $\mathcal{Q}$ there is a family $\mathcal{G}$ with property $\mathcal{R}$," we say that $\mathcal{P}^r$ is the property "for every family $\mathcal{F}$ *of regular cardinality* with property $\mathcal{Q}$, there is a family $\mathcal{G}$ with property $\mathcal{R}$." Property $\mathcal{P}^r$ is called "the property $\mathcal{P}$ with the restriction to regular cardinality."

It should be noted that the property $\mathcal{P}^r$ may depend on the exact statement of $\mathcal{P}$. For example consider the following:

$\mathcal{P}$: Every open cover has a countable subcover,

$\mathcal{P}^r$: Every open cover of regular cardinality has a countable subcover.

$\mathcal{Q}$: Every uncountable open cover has a subcover of strictly smaller cardinality.

$\mathcal{Q}^r$: Every uncountable open cover of regular cardinality has a subcover of strictly smaller cardinality.

Property $\mathcal{P}$ is the standard definition of the Lindelöf property, and Q is clearly equivalent to $\mathcal{P}$. But $\mathcal{P}^r$ and $\mathcal{Q}^r$ are not equivalent. It is known that $\mathcal{P}^r$ is equivalent to the Lindelöf property (see Theorem 2 in [V]; $\mathcal{P}^r$ is the property which is there called $S[\omega_1,\infty]$). Property $\mathcal{Q}^r$, however is equivalent to the property called "finally compact in the sense of complete accumulation points" (by a theorem of Alexandroff and Urysohn [AU]) which is strictly weaker than the Lindelöf property (see [M]).

In [WW], Worrell and Wicke continued a line of investigation started in [AU]. They considered the property $\mathcal{P}^r$, where $\mathcal{P}$ is the standard definition of weakly $\delta\Theta$-refinable, and they proved that every countably compact space with property $\mathcal{P}^r$ is compact. Their proof easily adapts to prove that every countably compact ultrapure^r space is compact. For astral and pure, however, we have

4.4. *Theorem. The following statements are independent of and consistent with* ZFC.

1. *Every countably compact pure*^{r} T_2*-space is compact.*

2. *Every countably comapct astral*^{r} T_2*-space is compact.*

Proof. Since in a countably compact space, every maximal filter is astral, we have that (1) implies (2). If 2^{ω_1} is singular then both statements are false: Consider the space $X = \omega_1$ with the order topology. Let Y denote the discrete subspace of successor ordinals. If $\mathcal{A}$ is astral on X, the closed set $(X - Y)$ is in $\mathcal{A}$ because otherwise $u = \{A \cap Y: A \in \mathcal{A}\}$ is a non-principal, ω^+-complete ultra-filter on the set Y. Now, for each $H \subset Y$, we have that the closed set $H \cup (X - Y) \in \mathcal{A}$; so $|\mathcal{A}| = 2^{\omega_1}$. Thus $X = \omega_1$ is astral^r in the vacuous sense; so (1) and (2) fail. On the other hand under GCH both (1) and (2) are true: if (1) is not true then by Theorem 4.2(c) there exists a pure^r T_2-space X which is not pure; so there exists a maximal filter $\mathcal{F}$ on X with $|\mathcal{F}|$ a singular cardinal. By GCH, $|\mathcal{F}|$ is a singular, strong limit cardinal, but since X is T_2 this contradicts Theorem 2.5.

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