SPANS OF AN ODD TRIOD

by

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In this paper we solve a problem which has been raised
in connection with some geometric aspects of span theory.
Although the spans of general metric spaces are mentioned,
the most interesting applications seem to belong in the
theory of continua.

1. Introduction

Generally speaking, the spans of an object are connected-
ness type analogues of its diameter. We follow the defini-
tions from [4]. Let X be a non-empty connected metric space.
The standard projections of the product $X \times X$ onto X are
denoted by p_1 and p_2, that is, $p_1(x,x') = x$ and $p_2(x,x') = x'$
for $(x,x') \in X \times X$. The *surjective span* $\sigma^*(X)$ of X is the
least upper bound of real numbers α such that there exist
non-empty connected sets $C_\alpha \subset X \times X$ with $\text{dist}(x,x') \geq \alpha$ for
$(x,x') \in C_\alpha$ and $p_1(C_\alpha) = p_2(C_\alpha) = X$. Relaxing the last con-
dition to $p_1(C_\alpha) = p_2(C_\alpha)$, or $p_2(C_\alpha) = X$, or $p_1(C_\alpha) \subset p_2(C_\alpha)$,
one obtains the definitions of the *span* $\sigma(X)$, the *surjective
semispan* $\sigma_0^*(X)$, and the *semispan* $\sigma_0(X)$ of X, respectively.
Hence

(1) $0 \leq \sigma^*(X) \leq \sigma(X) \leq \sigma_0(X) \leq \text{diam } X,$

(2) $0 \leq \sigma^*(X) \leq \sigma_0^*(X) \leq \sigma_0(X) \leq \text{diam } X.$

There is a conjecture (see [2], Problem 1) that arc-like
continua are the only continua with span zero. As of this
writing, it is still unsettled. Let us take a continuum X
with $\sigma(X) \neq 0$ and consider the ratio

$$\lambda = \frac{\sigma^*(X)}{\sigma(X)} .$$

By (1), we have $0 \leq \lambda \leq 1$. It has been conjectured that $\lambda \geq \frac{1}{2}$ whenever λ is defined (see [4], Problem 1), and a simple triod has been constructed [3] for which $\lambda = \frac{1}{2}$. There are many obvious examples of objects with $\lambda = 1$; for instance, any circle has the surjective span equal to its diameter. A natural question then can be asked whether or not the ratio λ can take any value from the interval $[\frac{1}{2}, 1]$. We answer this question in the affirmative (see Sections 2 and 3) by constructing certain simple triods in the Euclidean 3-space and calculating their spans. We do not know whether such examples exist on the plane.

2. Example

In this construction, the Euclidean 3-space R^3 with the ordinary Pythagorean distance d will be used. Given any number λ with $\frac{1}{2} < \lambda \leq 1$, we show the existence of a simple triod $T_\lambda \subset R^3$ such that

(3) $\sigma(T_\lambda) = \sigma_0(T_\lambda) = 1,$

(4) $\sigma^*(T_\lambda) = \sigma_0^*(T_\lambda) = \lambda.$

For two points a, b $\in R^3$, we denote by $\overline{ab}$ the straight line interval having a and b as endpoints. The simple triod T_λ is the union of three polygonal arcs each two of which have exactly one point in common, the origin $v = (0,0,0)$. Namely, we let

$$a_1 = (-1,0,0), \qquad a_2 = (0,1,0), \qquad a_3 = (1,0,0),$$
$$b_1 = (-\tfrac{1}{2},0,\sqrt{\lambda^2-\tfrac{1}{4}}), \qquad b_2 = (0,\tfrac{1}{2},\sqrt{\lambda^2-\tfrac{1}{4}}),$$
$$b_3 = (\tfrac{1}{2},0,\sqrt{\lambda^2-\tfrac{1}{4}}), \qquad a_4 = (\tfrac{3}{4},0,\tfrac{1}{4}),$$

and $A_i = \overline{a_i v}$ ($i = 1,2,3$). We define

$$T = A_1 \cup A_2 \cup A_3, \quad B = \overline{a_1 b_1} \cup \overline{b_1 b_2} \cup \overline{b_2 b_3} \cup \overline{b_3 a_4},$$

and $T_\lambda = T \cup B$. The three polygonal arcs forming the simple triod T_λ are $A_1 \cup B$, A_2 and A_3. To evaluate the spans of T_λ, we prove the following four claims.

Claim 1. $1 \leq \sigma(T_\lambda)$. Notice that T is a simple triod, contained in T_λ, and formed by three straight line intervals A_i ($i = 1,2,3$). Each of them has the endpoint a_i of distance 1 from the union of the other two intervals. It follows that $\sigma(T) \geq 1$ and $\sigma(T_\lambda) \geq \sigma(T)$ (see [5], p. 210, and [4], p. 36), whence $\sigma(T_\lambda) \geq 1$.

Claim 2. $\sigma_0(T_\lambda) \leq 1$. Let $C \subset T_\lambda \times T_\lambda$ be a non-empty connected set such that $p_1(C) \subset p_2(C)$. Then $p_2(C)$ is a connected subset of T_λ. The vertex v cuts the simple triod T_λ into three components which are subsets of the three polygonal arcs forming T_λ. If $v \notin p_2(C)$, the connected set $p_2(C)$ lies on one of these arcs and so does $p_1(C)$. The semispan of each arc is zero (see [4], p. 36), whence C contains points, (q,q'), with $d(q,q')$ arbitrarily small; in particular, $d(q,q') \leq 1$ for some $(q,q') \in C$. If $v \in p_2(C)$, then there exists a point $q_0 \in T_\lambda$ such that $(q_0,v) \in C$. Since $d(a_i,v) = 1$ ($i = 1,2,3$), $d(b_i,v) = \lambda \leq 1$ ($i = 1,2,3$) and $d(a_4,v) = 10^{\frac{1}{2}}/4 < 1$, all points of the intervals whose union is T_λ are of distance from v not exceeding 1. Thus $d(q_0,v) \leq 1$. In both cases, there does not exist any number $\alpha > 1$ such that $d(q,q') \geq \alpha$ for $(q,q') \in C$. It follows that $\sigma_0(T_\lambda) \leq 1$.

We observe that Claims 1 and 2, when combined with inequalities (1), yield equalities (3).

Claim 3. $\lambda \leq \sigma^*(T_\lambda)$. The set $D \subset T_\lambda \times T_\lambda$ defined by the formula

$$D = [(A_1 \cup A_3 \cup B) \times \{a_2\}] \cup [\{a_3\} \times (A_1 \cup A_2)] \cup$$
$$\cup [(A_2 \cup A_3) \times \{a_1\}] \cup [\{a_2\} \times (A_1 \cup A_3 \cup B)]$$

satisfies the condition that $p_1(C) = p_2(C) = T_\lambda$. As defined, D is the union of four connected sets, actually arcs (shown in the brackets), each of which meets the succeeding set. Indeed, (a_3, a_2) belongs to the first two sets, (a_3, a_1) belongs to the next two, and (a_2, a_1) belongs to the last two. So, D is connected. Clearly, $d(q,q') \geq 1$ for (q,q') belonging to either of the middle sets, as well as for $q \in A_1 \cup A_3$ and $q' = a_2$. To complete the proof of Claim 3, we need only to show that

(5) $\lambda \leq d(q, a_2)$ $(q \in B)$.

Since the points a_1, b_1, b_3, a_4 all belong to the plane $y = 0$ and the distance from a_2 to this plane is 1, we have $d(q, a_2) \geq 1 \geq \lambda$ for $q \in \overline{a_1 b_1} \cup \overline{b_3 a_4}$. The two remaining intervals of B to consider are $\overline{b_1 b_2}$ and $\overline{b_2 b_3}$. The plane $x + y - \frac{1}{2} = 0$ passes through the point b_2, is perpendicular to the interval $\overline{b_1 b_2}$, and cuts the space R^3 between the points b_1 and a_2. Consequently, the angle formed by the intervals $\overline{b_1 b_2}$ and $\overline{b_2 a_2}$ is greater than 90°. But $d(b_2, a_2) = \lambda$, which implies that $d(q, a_2) \geq \lambda$ for $q \in \overline{b_1 b_2}$. The reflection of R^3 with respect to the plane $x = 0$ transforms $\overline{b_1 b_2}$ into $\overline{b_3 b_2}$ and keeps the point a_2 fixed. Hence, $d(q, a_2) \geq \lambda$ for $q \in \overline{b_2 b_3}$. This establishes (5), and Claim 3 is proved.

Now, let us denote by $\pi: R^3 \to R$ the standard projection of R^3 onto the x-axis, that is, $\pi(x,y,z) = x$ for $(x,y,z) \in R^3$. We want to show that

(6)   if $q \in T_\lambda$, $q' \in B$ and $\pi(q) = \pi(q')$, then

        $d(q,q') \leq \lambda$.

Let q,q' be such points and $q \neq q'$. Then $-1 \leq \pi(q) \leq 1$. If $\pi(q) = 0$, we have $q' = b_2$ and $q \in A_2$. Since $d(a_2,b_2) = d(v,b_2) = \lambda$, all points of the interval $A_2 = \overline{a_2 v}$ are of distance from b_2 not exceeding λ, so that $d(q,q') \leq \lambda$. If $\pi(q) \neq 0$, then $q \notin A_2$. Note that π is one-to-one on B, whence $q \notin B$. We obtain $q \in A_1 \cup A_3$, and thus q is a point of the x-axis. Consequently, since $\pi(q) = \pi(q')$, the inequality $d(q,q') \leq \lambda$ is equivalent to the condition that the point q' belongs to the closed circular cylinder Q of radius λ around the x-axis. The points a_1, b_1, b_2, b_3, a_4 have distances from the x-axis equal to

$$0, \; \sqrt{\lambda^2 - \tfrac{1}{4}}, \; \lambda, \; \sqrt{\lambda^2 - \tfrac{1}{4}}, \; \tfrac{1}{4},$$

respectively, and therefore all these five points belong to Q. But Q being convex, it also contains all points of the intervals joining any of them. In particular, $B \subset Q$ which gives $q' \in Q$, and the proof of (6) is complete.

Claim 4. $\sigma_0^*(T_\lambda) \leq \lambda$. Suppose on the contrary that $\sigma_0^*(T_\lambda) > \lambda$. There exists a number $\alpha_0 > \lambda$ and a connected set $C_{\alpha_0} \subset T_\lambda \times T_\lambda$ with $d(q,q') \geq \alpha_0$ for $(q,q') \in C_{\alpha_0}$ and $p_2(C_{\alpha_0}) = T_\lambda$. The closure E of C_{α_0} in $T_\lambda \times T_\lambda$ is a continuum, and we also have $d(q,q') \geq \alpha_0 > \lambda$ for $(q,q') \in E$ and $p_2(E) = T_\lambda$. In particular, $a_4 \in p_2(E)$ which means that

there exists a point $w \in T_\lambda$ such that $(w,a_4) \in E$. We note

that $-1 \leq \pi(w) \leq 1$, $\pi(a_4) = \frac{3}{4}$, and distinguish two cases:

(i) $\pi(w) \leq \frac{3}{4}$, or (ii) $\pi(w) > \frac{3}{4}$.

If (i) holds, we consider the set $F = p_2^{-1}(B) \cap E$. It

is a proper subset of E because B is a proper subset of

$T_\lambda = p_2(E)$. Moreover, F is closed in E and $(w,a_4) \in F$.

Let K be the component of F containing (w,a_4). There

exists a point $(u,u') \in K$ which belongs to the boundary

of F in E (see [1], p. 172). Its image u' under the pro-

jection p_2 must belong to the boundary of B in T_λ, which is

the singleton $\{a_1\}$. Hence $u' = a_1$, and the continuum K

contains both (u,a_1) and (w,a_4). Since

$$\pi p_1(u,a_1) = \pi(u) \geq -1 = \pi(a_1) = \pi p_2(u,a_1),$$

$$\pi p_1(w,a_4) = \pi(w) \leq \frac{3}{4} = \pi(a_4) = \pi p_2(w,a_4),$$

there exists a point $(q,q') \in K$ such that $\pi p_1(q,q') = $

$\pi p_2(q,q')$, that is, $\pi(q) = \pi(q')$. Also, $q' \in p_2(K) \subset$

$p_2(F) \subset B$ and $(q,q') \in F \subset E$, whence $d(q,q') > \lambda$ which

contradicts (6).

If (ii) holds, we have $w \notin B$ because $\pi(B) = [-1,\frac{3}{4}]$,

and $w \notin A_1 \cup A_2$ because $\pi(A_1 \cup A_2) = [-1,0]$. Thus $w \in A_3$,

and $w \in \overline{w_0 a_3}$, where $w_0 = (\frac{3}{4},0,0)$. But $d(w_0,a_4) = \frac{1}{4}$ and

$d(a_3,a_4) = 2^{\frac{1}{2}}/4 < \frac{1}{2} < \lambda$, whence $d(w,a_4) < \lambda$. However,

$(w,a_4) \in E$ implies $d(w,a_4) > \lambda$, a contradiction.

The contradictions obtained in both cases (i) and (ii)

complete the proof of Claim 4. Note that Claims 3 and 4

combined with (2) give (4).

3. Comment

The construction of the simple triod T_λ can be extended to cover the limit value $\lambda = \frac{1}{2}$ if some modifications are made in the sub-triod T. For a sufficiently small number $\varepsilon > 0$, let

$$c_1 = (-\tfrac{1}{2},-\varepsilon,0), \qquad c_2 = (0,\tfrac{1}{2},-\varepsilon), \qquad c_3 = (\tfrac{1}{2},-\varepsilon,0),$$

and $A_i' = \overline{a_i c_i} \cup \overline{c_i v}$ $(i = 1,2,3)$. Keeping the same definition of B for $\lambda = \frac{1}{2}$, and taking $T_{\frac{1}{2}} = A_1' \cup A_2' \cup A_3' \cup B$, one has a simple triod $T_{\frac{1}{2}}$ which satisfies (3) and (4) for $\lambda = \frac{1}{2}$. The proof is quite analogous to that presented in Section 2. We do not attempt to make this description more precise since a simple triod with identical spans is already known [3].

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