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## SPANS OF AN ODD TRIOD

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## SPANS OF AN ODD TRIOD

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In this paper we solve a problem which has been raised in connection with some geometric aspects of span theory. Although the spans of general metric spaces are mentioned, the most interesting applications seem to belong in the theory of continua.

## 1. Introduction

Generally speaking, the spans of an object are connectedness type analogues of its diameter. We follow the definitions from [4]. Let $X$ be a non-empty connected metric space. The standard projections of the product $X \times X$ onto $X$ are denoted by $p_{1}$ and $p_{2}$, that is, $p_{1}\left(x, x^{\prime}\right)=x$ and $p_{2}\left(x, x^{\prime}\right)=x^{\prime}$ for $\left(\mathrm{x}, \mathrm{X}^{\prime}\right) \in \mathrm{X} \times \mathrm{X}$. The surjective $\operatorname{span} \sigma^{*}(\mathrm{X})$ of X is the least upper bound of real numbers $\alpha$ such that there exist non-empty connected sets $C_{\alpha} \subset X \times X$ with dist $\left(x, x^{\prime}\right) \geq \alpha$ for $\left(x, x^{\prime}\right) \in C_{\alpha}$ and $p_{1}\left(C_{\alpha}\right)=p_{2}\left(C_{\alpha}\right)=x$. Relaxing the last condition to $\mathrm{p}_{1}\left(\mathrm{C}_{\alpha}\right)=\mathrm{p}_{2}\left(\mathrm{C}_{\alpha}\right)$, or $\mathrm{p}_{2}\left(\mathrm{C}_{\alpha}\right)=\mathrm{X}$, or $\mathrm{p}_{1}\left(\mathrm{C}_{\alpha}\right) \subset \mathrm{p}_{2}\left(\mathrm{C}_{\alpha}\right)$, one obtains the definitions of the span $\sigma(\mathrm{X})$, the surjective semispan $\sigma_{0}^{*}(X)$, and the semispan $\sigma_{0}(X)$ of $X$, respectively. Hence

$$
\begin{equation*}
0 \leq \sigma^{*}(X) \leq \sigma(X) \leq \sigma_{0}(X) \leq \operatorname{diam} X, \tag{l}
\end{equation*}
$$

$$
\begin{equation*}
0 \leq \sigma^{*}(X) \leq \sigma_{0}^{*}(X) \leq \sigma_{0}(X) \leq \operatorname{diam} X \tag{2}
\end{equation*}
$$

There is a conjecture (see [2], Problem 1) that arc-like continua are the only continua with span zero. As of this writing, it is still unsettled. Let us take a continuum $X$ with $\sigma(X) \neq 0$ and consider the ratio

$$
\lambda=\frac{\sigma^{*}(X)}{\sigma(X)}
$$

By (1), we have $0 \leq \lambda \leq 1$. It has been conjectured that $\lambda \geq \frac{1}{2}$ whenever $\lambda$ is defined (see [4], Problem 1), and a simple triod has been constructed [3] for which $\lambda=\frac{1}{2}$. There are many obvious examples of objects with $\lambda=1$; for instance, any circle has the surjective span equal to its diameter. A natural question then can be asked whether or not the ratio $\lambda$ can take any value from the interval $\left[\frac{1}{2}, 1\right]$. We answer this question in the affirmative (see Sections 2 and 3) by constructing certain simple triods in the Euclidean 3-space and calculating their spans. We do not know whether such examples exist on the plane.

## 2. Example

In this construction, the Euclidean 3-space $\mathrm{R}^{3}$ with the ordinary Pythagorean distance $d$ will be used. Given any number $\lambda$ with $\frac{1}{2}<\lambda \leq 1$, we show the existence of a simple triod $T_{\lambda} \subset R^{3}$ such that

$$
\begin{align*}
& \sigma\left(\mathrm{T}_{\lambda}\right)=\sigma_{0}\left(\mathrm{~T}_{\lambda}\right)=1,  \tag{3}\\
& \sigma^{\star}\left(\mathrm{T}_{\lambda}\right)=\sigma_{0}^{*}\left(\mathrm{~T}_{\lambda}\right)=\lambda . \tag{4}
\end{align*}
$$

For two points $a, b \in R^{3}$, we denote $b y \overline{a b}$ the straight line interval having $a$ and $b$ as endpoints. The simple triod $T_{\lambda}$ is the union of three polygonal arcs each two of which have exactly one point in common, the origin $v=(0,0,0)$. Namely, we let

$$
\begin{aligned}
a_{1} & =(-1,0,0), \quad a_{2}=(0,1,0), \quad a_{3}=(1,0,0), \\
b_{1} & =\left(-\frac{1}{2}, 0, \sqrt{\lambda^{2}-\frac{1}{4}}\right), \quad b_{2}=\left(0, \frac{1}{2}, \sqrt{\lambda^{2}-\frac{1}{4}}\right), \\
b_{3} & =\left(\frac{1}{2}, 0, \sqrt{\lambda^{2}-\frac{1}{4}}\right), \quad a_{4}=\left(\frac{3}{4}, 0, \frac{1}{4}\right), \\
\text { and } A_{i}=\frac{a_{i}}{} \mathrm{v}(i & =1,2,3) . \quad \text { We define }
\end{aligned}
$$

$$
T=A_{1} \cup A_{2} \cup A_{3}, \quad B=\overline{a_{1} b_{1}} \cup \overline{b_{1} b_{2}} \cup \overline{b_{2} b_{3}} \cup \overline{b_{3} a_{4}}
$$

and $T_{\lambda}=T U B$. The three polygonal arcs forming the simple triod $T{ }_{\lambda}$ are $A_{1} \cup B, A_{2}$ and $A_{3}$. To evaluate the spans of $T_{\lambda}$, we prove the following four claims.

Claim 1. $1 \leq \sigma\left(T_{\lambda}\right)$. Notice that $T$ is a simple triod, contained in $T_{\lambda}$, and formed by three straight line intervals $A_{i}(i=1,2,3)$. Each of them has the endpoint $a_{i}$ of distance 1 from the union of the other two intervals. It follows that $\sigma(T) \geq 1$ and $\sigma\left(T_{\lambda}\right) \geq \sigma(T)$ (see [5], p. 210, and [4], p. 36), whence $\sigma\left(T_{\lambda}\right) \geq 1$.

Claim 2. $\sigma_{0}\left(T_{\lambda}\right) \leq 1$. Let $C \subset T_{\lambda} \times T_{\lambda}$ be a non-empty connected set such that $p_{1}(C) \subset p_{2}(C)$. Then $p_{2}(C)$ is a connected subset of $T_{\lambda}$. The vertex $v$ cuts the simple triod $T_{\lambda}$ into three components which are subsets of the three polygonal arcs forming $T_{\lambda}$. If $v \notin p_{2}(C)$, the connected set $P_{2}(C)$ lies on one of these arcs and so does $P_{1}(C)$. The semispan of each arc is zero (see [4], p. 36), whence C contains points, ( $q, q^{\prime}$ ), with $d\left(q, q^{\prime}\right)$ arbitrarily small; in particular, $d\left(q, q^{\prime}\right) \leq l$ for some $\left(q, q^{\prime}\right) \in C . \quad$ If $v \in \mathrm{p}_{2}(\mathrm{C})$, then there exists a point $\mathrm{q}_{0} \in \mathrm{~T}_{\lambda}$ such that $\left(q_{0}, v\right) \in C . \quad$ Since $d\left(a_{i}, v\right)=1(i=1,2,3), d\left(b_{i}, v\right)=\lambda \leq 1$ ( $i=1,2,3$ ) and $d\left(a_{4}, v\right)=10^{\frac{1}{2}} / 4<1$, all points of the intervals whose union is $T_{\lambda}$ are of distance from $v$ not exceeding 1. Thus $d\left(q_{0}, v\right) \leq 1$. In both cases, there does not exist any number $\alpha>1$ such that $d\left(q, q^{\prime}\right) \geq \alpha$ for $\left(q, q^{\prime}\right) \in C . \quad$ It follows that $\sigma_{0}\left(T_{\lambda}\right) \leq 1$.

We observe that Claims 1 and 2, when combined with inequalities (1), yield equalities (3).

Claim 3. $\lambda \leq \sigma^{*}\left(\mathrm{~T}_{\lambda}\right)$. The set $\mathrm{D} \subset \mathrm{T}_{\lambda} \times \mathrm{T}_{\lambda}$ defined by the formula

$$
\begin{aligned}
D= & {\left[\left(A_{1} \cup A_{3} \cup B\right) \times\left\{a_{2}\right\}\right] \cup\left[\left\{a_{3}\right\} \times\left(A_{1} \cup A_{2}\right)\right] \cup } \\
& \cup\left[\left(A_{2} \cup A_{3}\right) \times\left\{a_{1}\right\}\right] \cup\left[\left\{a_{2}\right\} \times\left(A_{1} \cup A_{3} \cup B\right)\right]
\end{aligned}
$$

satisfies the condition that $p_{1}(C)=p_{2}(C)=T_{\lambda}$. As defined, $D$ is the union of four connected sets, actually arcs (shown in the brackets), each of which meets the succeeding set. Indeed, $\left(a_{3}, a_{2}\right)$ belongs to the first two sets, $\left(a_{3}, a_{1}\right)$ belongs to the next two, and ( $a_{2}, a_{1}$ ) belongs to the last two. So, D is connected. Clearly, $d\left(q, q^{\prime}\right) \geq 1$ for ( $q, q^{\prime}$ ) belonging to either of the middle sets, as well as for $q \in A_{1} \cup A_{3}$ and $q^{\prime}=a_{2}$. To complete the proof of Claim 3, we need only to show that

$$
\begin{equation*}
\lambda \leq d\left(q, a_{2}\right) \quad(q \in B) \tag{5}
\end{equation*}
$$

Since the points $a_{1}, b_{1}, b_{3}, a_{4}$ all belong to the plane $y=0$ and the distance from $a_{2}$ to this plane is 1 , we have $d\left(q, a_{2}\right) \geq 1 \geq \lambda$ for $q \in \overline{a_{1} b_{1}} \cup \overline{b_{3} a_{4}}$. The two remaining intervals of $B$ to consider are $\overline{b_{1} b_{2}}$ and $\overline{b_{2} b_{3}}$. The plane $x+y-\frac{1}{2}=0$ passes through the point $b_{2}$, is perpendicular to the interval $\overline{b_{1} b_{2}}$, and cuts the space $\mathrm{R}^{3}$ between the points $b_{1}$ and $a_{2}$. Consequently, the angle formed by the intervals $\overline{\mathrm{b}_{1} \mathrm{~b}_{2}}$ and $\overline{\mathrm{b}_{2} \mathrm{a}_{2}}$ is greater than $90^{\circ}$. But $d\left(b_{2}, a_{2}\right)=\lambda$, which implies that $d\left(q, a_{2}\right) \geq \lambda$ for $q \in \overline{b_{1} b_{2}}$. The reflection of $R^{3}$ with respect to the plane $x=0$ transforms $\overline{b_{1} b_{2}}$ into $\overline{b_{3} b_{2}}$ and keeps the point $a_{2}$ fixed. Hence, $d\left(q, a_{2}\right) \geq \lambda$ for $q \in \overline{b_{2} b_{3}}$. This establishes (5), and Claim 3 is proved.

Now, let us denote by $\pi: R^{3} \rightarrow R$ the standard projection of $R^{3}$ onto the $x$-axis, that is, $\pi(x, y, z)=x$ for $(x, y, z) \in R^{3}$. We want to show that

$$
\begin{align*}
& \text { if } q \in T_{\lambda}, q^{\prime} \in B \text { and } \pi(q)=\pi\left(q^{\prime}\right) \text {, then }  \tag{6}\\
& d\left(q, q^{\prime}\right) \leq \lambda \text {. }
\end{align*}
$$

Let $q, q^{\prime}$ be such points and $q \neq q^{\prime}$. Then $-1 \leq \pi(q) \leq 1$. If $\pi(q)=0$, we have $q^{\prime}=b_{2}$ and $q \in A_{2}$. Since $d\left(a_{2}, b_{2}\right)=$ $d\left(v, b_{2}\right)=\lambda$, all points of the interval $A_{2}=\overline{a_{2} v}$ are of distance from $b_{2}$ not exceeding $\lambda$, so that $d\left(q, q^{\prime}\right) \leq \lambda$. If $\pi(q) \neq 0$, then $q \notin A_{2}$. Note that $\pi$ is one-to-one on $B$, whence $q \notin B$. We obtain $q \in A_{1} \cup A_{3}$, and thus $q$ is a point of the $x$-axis. Consequently, since $\pi(q)=\pi\left(q^{\prime}\right)$, the inequality $d\left(q, q^{\prime}\right) \leq \lambda$ is equivalent to the condition that the point $q^{\prime}$ belongs to the closed circular cylinder $Q$ of radius $\lambda$ around the $x$-axis. The points $a_{1}, b_{1}, b_{2}, b_{3}, a_{4}$ have distances from the $x$-axis equal to

$$
0, \sqrt{\lambda^{2}-\frac{1}{4}}, \lambda, \sqrt{\lambda^{2}-\frac{1}{4}}, \frac{1}{4},
$$

respectively, and therefore all these five points belong to Q. But $Q$ being convex, it also contains all points of the intervals joining any of them. In particular, $B \subset Q$ which gives $q^{\prime} \in Q$, and the proof of (6) is complete.

Claim 4. $\sigma_{0}^{*}\left(T_{\lambda}\right) \leq \lambda$. Suppose on the contrary that $\sigma_{0}^{*}\left(T_{\lambda}\right)>\lambda$. There exists a number $\alpha_{0}>\lambda$ and a connected set $C_{\alpha_{0}} \subset T_{\lambda} \times T_{\lambda}$ with $d\left(q, q^{\prime}\right) \geq \alpha_{0}$ for $\left(q, q^{\prime}\right) \in C_{\alpha_{0}}$ and $\mathrm{p}_{2}\left(\mathrm{C}_{\alpha_{0}}\right)=\mathrm{T}_{\lambda}$. The closure E of $\mathrm{C}_{\alpha_{0}}$ in $\mathrm{T}_{\lambda} \times \mathrm{T}_{\lambda}$ is a continuum, and we also have $d\left(q, q^{\prime}\right) \geq \alpha_{0}>\lambda$ for $\left(q, q^{\prime}\right) \in E$ and $p_{2}(E)=T_{\lambda}$. In particular, $a_{4} \in p_{2}(E)$ which means that
there exists a point $w \in T_{\lambda}$ such that $\left(w, a_{4}\right) \in E$. We note that $-1 \leq \pi(w) \leq 1, \pi\left(a_{4}\right)=\frac{3}{4}$, and distinguish two cases: (i) $\pi(w) \leq \frac{3}{4}$, or (ii) $\pi(w)>\frac{3}{4}$.

If (i) holds, we consider the set $F=P_{2}^{-1}(B) \cap E . \quad$ It is a proper subset of $E$ because $B$ is a proper subset of $T_{\lambda}=p_{2}(E)$. Moreover, $F$ is closed in $E$ and ( $w, a_{4}$ ) $\in F$. Let $K$ be the component of $F$ containing ( $w, a_{4}$ ). There exists a point ( $u, u^{\prime}$ ) $\in K$ which belongs to the boundary of $F$ in $E$ (see [1], p. 172). Its image $u$ ' under the projection $P_{2}$ must belong to the boundary of $B$ in $T_{\lambda}$, which is the singleton $\left\{a_{1}\right\}$. Hence $u^{\prime}=a_{1}$, and the continuum $k$ contains both ( $u, a_{1}$ ) and ( $w, a_{4}$ ). Since

$$
\begin{aligned}
& \pi p_{1}\left(u, a_{1}\right)=\pi(u) \geq-1=\pi\left(a_{1}\right)=\pi p_{2}\left(u, a_{1}\right), \\
& \pi p_{1}\left(w, a_{4}\right)=\pi(w) \leq \frac{3}{4}=\pi\left(a_{4}\right)=\pi p_{2}\left(w, a_{4}\right)
\end{aligned}
$$

there exists a point $\left(q, q^{\prime}\right) \in K$ such that $\pi p_{1}\left(q, q^{\prime}\right)=$ $\pi p_{2}\left(q, q^{\prime}\right)$, that is, $\pi(q)=\pi\left(q^{\prime}\right)$. Also, $q^{\prime} \in p_{2}(K) \subset$ $p_{2}(F) \subset B$ and $\left(q, q^{\prime}\right) \in F \subset E$, whence $d\left(q, q^{\prime}\right)>\lambda$ which contradicts (6).

If (ii) holds, we have $w \notin B$ because $\pi(B)=\left[-1, \frac{3}{4}\right]$, and $w \notin A_{1} \cup A_{2}$ because $\pi\left(A_{1} \cup A_{2}\right)=[-1,0]$. Thus $w \in A_{3}$, and $w \in \overline{w_{0} a_{3}}$, where $w_{0}=\left(\frac{3}{4}, 0,0\right)$. But $d\left(w_{0}, a_{4}\right)=4$ and $d\left(a_{3}, a_{4}\right)=2^{\frac{1}{2}} / 4<\frac{1}{2}<\lambda$, whence $d\left(w, a_{4}\right)<\lambda$. However, $\left(w, a_{4}\right) \in E$ implies $d\left(w, a_{4}\right)>\lambda$, a contradiction.

The contradictions obtained in both cases (i) and (ii) complete the proof of Claim 4. Note that Claims 3 and 4 combined with (2) give (4).

## 3. Comment

The construction of the simple triod $\mathrm{T}_{\lambda}$ can be extended to cover the limit value $\lambda=\frac{1}{2}$ if some modifications are made in the sub-triod T. For a sufficiently small number $\varepsilon>0$, let
$c_{1}=\left(-\frac{1}{2},-\varepsilon, 0\right), \quad c_{2}=\left(0, \frac{1}{2},-\varepsilon\right), \quad c_{3}=\left(\frac{1}{2},-\varepsilon, 0\right)$, and $A_{i}=\overline{a_{i} c_{i}} \cup \overline{c_{i} v}(i=1,2,3) . \quad$ Keeping the same definition of $B$ for $\lambda=\frac{1}{2}$, and taking $T_{\frac{1}{2}}=A_{1}^{\prime} \cup A_{2}^{\prime} \cup A_{3}^{\prime} \cup B$, one has a simple triod $T_{\frac{1}{2}}$ which satisfies (3) and (4) for $\lambda=\frac{1}{2}$. The proof is quite analogous to that presented in Section 2. We do not attempt to make this description more precise since a simple triod with identical spans is already known [3].

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