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# TOPOLOGY PROCEEDINGS



Volume 8, 1983

Pages 361–394

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<http://topology.auburn.edu/tp/>

## CONTINUUM THEORY PROBLEMS

by

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### Topology Proceedings

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**ISSN:** 0146-4124

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## CONTINUUM THEORY PROBLEMS

**Wayne Lewis**

The problems listed below have come from a number of sources. Some were posed at the Texas Topology Symposium 1980 in Austin, some at the American Mathematical Society meeting in Baton Rouge in 1982, some at the Topology Conference in Houston in 1983, some at discussions at the University of Florida in 1982, and some at the International Congress of Mathematicians in Warsaw in 1983. Some are classical, while others are more recent or primarily of technical interest. Preliminary versions of subsets of this list have been circulated, and an attempt has been made to verify the accuracy of the statements of the questions, comments, and references given. In many cases, variations on a given question have been asked by many people on diverse occasions. Thus the version presented here should not be considered definitive. Any errors or additions which are brought to my attention will be noted at a later date.

The division of the questions into categories is only intended as a rough guide, and many questions could properly be placed in more than one category. A number of these questions have appeared in the University of Houston Problem Book (UHPB), a good reference for further problems. Assistance in compiling earlier versions of subsets of this list was provided by Bellamy, Brechner, Heath, and Mayer.

### Chainable Continua

1. (*Brechner, Lewis, Toledo*) Can a chainable continuum admit two non-conjugate homeomorphisms of period  $n$  with the same fixed point set?

[Earlier (*Brechner*): Are every two period  $n$  homeomorphisms of the pseudo-arc conjugate? Lewis has since shown that the pseudo-arc admits homeomorphisms of every period, and Toledo has shown that it admits such homeomorphisms with nondegenerate fixed point sets.]

2. (*Brechner*) Classify, up to conjugacy, the periodic homeomorphisms of the pseudo-arc?

3. (*Anderson*) Does every Cantor group act effectively on the pseudo-arc?

[Lewis has shown that every inverse limit of finite solvable groups acts effectively on the pseudo-arc.]

4. (*Nadler*) Does the pseudo-arc have the complete invariance property?

[A continuum  $X$  has the *complete invariance property* if every non-empty closed subset of  $X$  is the fixed point set of some continuous self-map of  $X$ . Martin and Nadler have shown that every two-point set is a fixed point set for some continuous self-map of the pseudo-arc. Cornette has shown that every subcontinuum of the pseudo-arc is a retract. Toledo has shown that every subcontinuum is the fixed

point set of a periodic homeomorphism. Lewis has shown that there are proper subsets of the pseudo-arc with non-empty interior which are the fixed point sets of homeomorphisms.]

5. (*Brechner and Lewis*) Do there exist stable homeomorphisms of the pseudo-arc which are extendable (or essentially extendable) to the plane? How many, up to conjugacy?

[This is a rewording of a question earlier posed by Brachner. Lewis has shown that there are non-identity stable homeomorphisms of the pseudo-arc.]

6. (*Brechner*) Let  $M$  be a particular embedding of the pseudo-arc in the plane, and let  $G$  be the group of extendable homeomorphisms of  $M$ . Does  $G$  characterize the embedding?

7. (*Lewis*) Are the periodic (resp. almost periodic, or pointwise-periodic) homeomorphisms dense in the group of homeomorphisms of the pseudo-arc?

[Conjecture is that answer is no. For each  $n \geq 2$ , the period  $n$  homeomorphisms do act transitively on the pseudo-arc.]

8. (*Brechner*) Does each periodic homeomorphism  $h$  of the pseudo-arc have a square root (i.e. a homeomorphism  $g$  such that  $g^2 = h$ )?

[It is known that some periodic homeomorphisms have an infinite sequence of  $p_i$ -roots, for any sequence  $\{p_i\}$  of positive integers.]

9. (*Toledo*) Can a pointwise-periodic, regular homeomorphism on a chainable (indecomposable), or tree-like (indecomposable) continuum, or the pseudo-arc, always be induced by square commuting diagrams on inverse systems of finite graphs?

[Fugate has shown that such homeomorphisms on chainable continua cannot always be induced by square commuting diagrams on inverse systems of arcs. Toledo has shown that periodic homeomorphisms of the pseudo-arc can always be induced by square commuting diagrams on finite graphs (not necessarily trees).]

10. (*Toledo*) Can a homeomorphism of a chainable continuum always be induced by square commuting diagrams on inverse systems of finite graphs?

[See remark after question 9.]

11. (*Duda*) Characterize chainability and/or circular chainability without using span.

[Oversteegen and Tymchatyn have a technical partial characterization, but a complete, useful, and satisfying characterization remains to be developed.]

12. (*Duda*) What additional condition(s) make the following statement true?

If  $X$  has only chainable proper subcontinua and  $(?)$ , then  $X$  is either chainable or circularly chainable.

[Ingram's examples show that an additional condition is needed. If  $X$  is decomposable, no additional condition is needed. If  $X$  is hereditarily indecomposable, then either homogeneity (or the existence of a  $G_\delta$  orbit under the action of its homeomorphism group) or weak chainability is a sufficient condition. Hereditary indecomposability alone is insufficient. Also (*Fugate*, UHPB 106): If  $M$  is tree-like and every proper subcontinuum of  $M$  is chainable, is  $M$  almost chainable?]

13. (*Fugate*, UHPB 104) If  $X$  is circularly chainable and  $f: X \rightarrow Y$  is open, then is  $Y$  either chainable or circularly chainable?

[Yes if  $Y$  is decomposable.]

14. (*Duda*) Can the following theorem be improved--say by dropping "hereditarily decomposable"?

*Theorem (Duda and Kell)* Let  $f: X \rightarrow Y$  be a finite-to-one open mapping of an hereditarily decomposable chainable continuum onto a  $T_2$  space. Then  $X = \bigcup_{j=1}^n K_j$  where each  $K_j$  is a continuum and  $f|_{\text{Int } K_j}$  is a homeomorphism.

15. (*Cook and Fugate*, UHPB 105) Suppose  $M$  is an atriodic one-dimensional continuum and  $G$  is an upper semi-continuous decomposition of  $M$  such that  $M/G$  and every element of  $G$  are chainable. Is  $M$  chainable?

[Michel Smith has shown that if "one-dimensional" is removed and " $M/G$  is an arc" is added to the hypothesis, then the answer is yes.]

[It follows from a result of Sher that even if  $M$  contains a triod, if  $M/G$  and every element of  $G$  are tree-like, then  $M$  is tree-like. If  $M$  is hereditarily indecomposable and  $G$  is continuous, the answer is yes.]

16. (*Mohler*) Is every weakly chainable, atriodic, tree-like continuum chainable?

[A positive answer would imply that the classification of homogeneous plane continua is complete.]

### Decompositions

17. (*Rogers*) Suppose  $G$  is a continuous decomposition of  $E^2$  into nonseparating continua. Must some element of  $G$  be hereditarily indecomposable? What if all of the decomposition elements are homeomorphic? Must some element have span zero? be chainable?

[This is a revision of a question by Mayer. Possibly related to this, Oversteegen and Mohler have recently shown that there exists an irreducible continuum  $X$  and an open, monotone map  $f: X \rightarrow [0,1]$  such that each nondegenerate

subcontinuum of  $X$  contains an arc, and so no nondegenerate  $f^{-1}(t)$  is hereditarily indecomposable. Oversteegen and Tymchatyn have shown that there must exist an  $f^{-1}(t)$  which contains arbitrarily small indecomposable subcontinua.]

18. (*Krasinkiewicz*, UHPB 158) Let  $X$  be a nondegenerate continuum such that there exists a continuous decomposition of the plane into elements homeomorphic to  $X$ . Must  $X$  be the pseudo-arc?

19. (*Mayer*) How many inequivalent embeddings of the pseudo-arc are to be found in the Lewis-Walsh decomposition of  $E^2$  into pseudo-arcs?

20. (*Ingram*) Does there exist a tree-like, non-chainable continuum  $M$  such that the plane contains uncountably many disjoint copies of  $M$ ? Is there a continuous collection of copies of  $M$  filling up the plane?

[Ingram has constructed an uncountable collection of disjoint, non-homeomorphic, tree-like, non-chainable continua in the plane.]

21. (*Lewis*) Is there a continuous decomposition of  $E^2$  into Ingram continua (not necessarily all homeomorphic)?

22. (*Lewis*) If  $M$  is an hereditarily equivalent or homogeneous, non-separating plane continuum, does there exist a continuous collection of continua, each homeomorphic to  $M$ , filling up the plane? Does the plane contain a (homogeneous) continuous circle of copies of  $M$ , as in the Jones Decomposition Theorem?



23. (*Lewis*) If  $X$  and  $Y$  are one-dimensional continua with continuous decompositions  $G$  and  $H$ , respectively, into pseudo-arcs such that  $X/G$  and  $Y/H$  are homeomorphic, then are  $X$  and  $Y$  homeomorphic?

[It follows from arguments of Lewis that if every element of  $G$  and  $H$  is a terminal continuum in  $X$  and  $Y$  respectively then  $X$  and  $Y$  are homeomorphic.]

24. (*Burgess*) Is there a continuous decomposition  $G$  of  $E^3$  into pseudo-arcs such that  $E^3/G \approx E^3$  and the pre-image of each one-dimensional continuum is one-dimensional? If so, is the pre-image of a homogeneous curve under such a decomposition itself homogeneous? Can this process produce any new homogeneous curves?

[It is known that for every one-dimensional continuum  $M$  there exists a one-dimensional continuum  $\tilde{M}$  with a continuous decomposition  $G$  into pseudo-arcs such that  $\tilde{M}/G \approx M$ . If  $M$  is homogeneous, then  $\tilde{M}$  can be constructed to be homogeneous. This method can produce new homogeneous continua.]

### Fixed Points

25. (*Bellamy*) Allowing singletons as degenerate indecomposable continua, are the following statements true?

a) Suppose  $X$  is a tree-like continuum and  $f: X \rightarrow X$  is continuous. Then there is an indecomposable subcontinuum  $W$  of  $X$  such that  $f(W) \subseteq W$ .

b) The same with hereditarily unicoherent replacing tree-like in the hypothesis.

[Bellamy has constructed a tree-like indecomposable continuum without the fixed point property. Manka has shown

that every  $\lambda$ -dendroid (hereditarily decomposable, hereditarily unicoherent continuum) has the fixed point property. Cook has shown that  $\lambda$ -dendroids are tree-like.]

26. (*Bellamy*) Suppose  $X$  is a tree-like continuum and every indecomposable subcontinuum has the fixed point property. Does  $X$  have the fixed point property?

27. (*Bellamy*) Suppose  $X$  is a tree-like continuum and  $f: X \rightarrow X$  is a function homotopic to the identity on  $X$ . Must  $f$  have a fixed point?

28. (*Bellamy*) Suppose  $X$  is a tree-like continuum. Does there exist  $\varepsilon > 0$  such that every self-map of  $X$  within  $\varepsilon$  of the identity has a fixed point?

29. (*Knaster*) Does every hereditarily indecomposable tree-like continuum have the fixed point property?

30. (*Cook*) Does every hereditarily equivalent continuum have the fixed point property?

[A continuum is *hereditarily equivalent* if it is homeomorphic with each of its nondegenerate subcontinua. Cook has shown that every nondegenerate hereditarily equivalent continuum other than the arc or pseudo-arc is hereditarily indecomposable and tree-like.]

31. (*Bellamy*) Suppose  $X$  is triod-like (or  $K$ -like for some fixed tree  $K$ ). Must  $X$  have the fixed point property?

32. (*Bellamy*) Does every inverse limit of real projective planes with homotopically essential bonding maps

have the fixed point property? for homeomorphisms?

33. Suppose  $X$  is a nonseparating plane continuum with each arc component dense. Is  $X$  an almost continuous retract of a disc?

[If  $X \subseteq D$ , a function  $f: D \rightarrow X$  is *almost* (quasi-) *continuous* if every neighborhood in  $D \times D$  (in  $D \times X$ ) of the graph of  $f$  contains the graph of a continuous function with domain  $D$ . Akis has shown that the disc with a spiral about its boundary is neither an almost continuous nor quasi-continuous retract of a disc.]

34. (Bellamy) Suppose  $f$  is a self-map of a tree-like continuum which commutes with some homeomorphism of period greater than one, or with every member of some nondegenerate group of homeomorphisms. Must  $f$  have a fixed point?

[Fugate has shown that if a compact group acts on a continuum, then all the homeomorphisms in the group have a common fixed point.]

35. (Edwards) Does every self-map (homeomorphism) of a tree-like continuum have a periodic point?

36. (Bellamy) Does every weakly chainable tree-like continuum have the fixed point property? What about tree-like continua which are the continuous image of circle-like continua?

37. (Rosenholts) Suppose  $f$  is a map from a non-separating plane continuum  $M$  to itself which is differentiable

(i.e.  $f$  can be extended to a neighborhood of  $M$  with partial derivatives existing). Must  $f$  have a fixed point?

38. (*Sternbach*, Scottish Book 107) Does every non-separating plane continuum have the fixed point property?

39. (*Bellamy*) Do each two commuting functions on a simple triod have a common incidence point?

40. (*Manka*) Let  $C$  be the composant with an endpoint in the simplest Knaster indecomposable continuum. Does  $C$  have the fixed point property?

[Also: If  $f: C \rightarrow C$  is continuous with non-compact image, is  $f$  onto? An affirmative answer gives an affirmative answer to the previous question.]

41. (*Oversteegen and Rogers*) Does the cone over  $X$  have the fixed point property, where  $X$  is the tree-like continuum without the fixed point property constructed by Oversteegen and Rogers?

42. (*Žysko*) Does there exist a continuum  $X$  with the fixed point property such that  $X \times P$  ( $P$  = pseudo-arc) does not have the fixed point property?

43. (*Gordh*) If  $X$  is an irreducible continuum and each tranch has the fixed point property, must  $X$  have the fixed point property?

[If  $X$  is an irreducible continuum such that each indecomposable subcontinuum of  $X$  is nowhere dense, then there exists a finest monotone map  $f: X \rightarrow [0,1]$ . Point-inverses

under  $f$  are nowhere dense subcontinua of  $X$  and are called the tranches of  $X$ .]

44. (*Bell*) Is there a map  $f: K \rightarrow K$ , where  $K$  is a continuum in  $\mathbb{R}^2$  and  $K$  is minimal with respect to  $f(K) \subset K$ , such that  $\text{Index}(f, K) = 0$ ?

[If  $g: A \rightarrow \mathbb{R}^{n+1}$  is a fixed point free map where  $A$  is an  $n$ -sphere in  $\mathbb{R}^{n+1}$ , then  $\text{Index}(g, A)$  is the degree of 
$$h(z) = \frac{g(z) - z}{\|g(z) - z\|}. \quad \text{If } K \text{ is a point-like continuum in } \mathbb{R}^n$$
 and  $f$  is a fixed-point free map  $f: \text{Bd}K \rightarrow \mathbb{R}^n$  then  $f$  has an extension to a map  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  that is fixed-point free on  $\mathbb{R}^n - K$ .  $\text{Index}(f, \text{Bd}K) = \text{Index}(F, B)$ , where  $B$  is any  $n - 1$  sphere in  $\mathbb{R}^n$  that surrounds  $K$ .]

45. (*Bell*) Let  $B$  be a point-like continuum in  $\mathbb{R}^n$ ,  $n > 2$ ,  $f: \text{Bd}(B) \rightarrow B$ , and  $\text{Index}(f, \text{Bd}B) = 0$ . Must there be a continuum  $K \subset \text{Bd}B$  such that  $K = f(K)$ ?

[Answer is no if there is a fixed-point free map on a point-like continuum  $K$ , where  $\text{Bd}K$  contains no invariant subcontinua.]

46. (*Minc*) Is there a planar continuum  $X$  and  $f: X \rightarrow X$  such that  $f$  induces the zero homomorphism on first Čech cohomology group and  $f$  is fixed point free?

#### Higher-Dimensional Problems

47. (*Anceł*) If  $f: S^2 \rightarrow \mathbb{R}^3$  is continuous and if  $U$  is the unbounded component of  $\mathbb{R}^3 - f(S^2)$  is  $f: S^2 \rightarrow \mathbb{R}^3 - U$  homotopically trivial in  $\mathbb{R}^3 - U$ ?

[The analogous result is true one dimension lower and false one dimension higher.]

48. (*Ance1*) If  $X$  is a cellular subset of  $\mathbb{R}^3$  is  $\pi_2(X) = 0$ ?

49. (*Burgess*) Is a 2-sphere  $S$  in  $S^3$  tame if it is homogeneously embedded (i.e. for each  $p, q$  in  $S$  there is a homeomorphism  $h: (S^3, S, p) \rightarrow (S^3, S, q)$ )?

50. (*Burgess*) Is a 2-sphere  $S$  in  $S^3$  tame if every homeomorphism of  $S$  onto itself can be extended to a homeomorphism of  $S^3$  onto itself?

51. (*Bing*) If  $S$  is a toroidal simple closed curve in  $E^3$  (i.e. an intersection of nested solid tori with small meridional cross-sections) such that over each arc  $A$  in  $S$  a singular fin can be raised, with no singularities on  $A$ , must  $S$  be tame?

[A fin is a disc which contains  $A$  as an arc on its boundary and is otherwise disjoint from  $S$ . It follows from a result of Burgess and Cannon that  $S$  is tame if the fin can always be chosen to be non-singular.]

52. (*Bing*) Is a simple closed curve  $S$  in  $E^3$  tame if it is isotopically homogeneous (i.e. for each  $p, q$  in  $S$  there is an ambient isotopy of  $E^3$ , leaving  $S$  invariant at each stage, with the 0-th level of the isotopy the identity and the last level a homeomorphism taking  $p$  to  $q$ )?

[Compare with dissertation and related work of Shilepsky.]

53. (*J. Heath, Jack Rogers*) If  $r: X \rightarrow Y$  is refinable and  $X \in \text{ANR}$ , must  $Y \in \text{ANR}$ ?

[A map  $r: X \rightarrow Y$  is refinable if for each  $\varepsilon > 0$  there is an  $\varepsilon$ -refinement, i.e. an  $\varepsilon$ -map  $g: X \rightarrow Y$  such that  $\text{dist}(g(x), r(x)) < \varepsilon$  for each  $x \in X$ . Heath and Kozłowski have shown: If  $X$  is finite dimensional, then  $Y$  must be an ANR if either (i) each  $r^{-1}(y)$  is locally connected, (ii) each  $r^{-1}(y)$  is nearly 1-movable, (iii) each  $r^{-1}(y)$  is approximately 1-connected, (iv)  $Y$  is  $\text{LC}^1$  at each point, or (v) there is a monotone  $\varepsilon$ -refinement of  $r$  for each  $\varepsilon > 0$ .]

54. (*J. Heath, Kozłowski*) If  $r: S^3 \rightarrow Y$  is refinable, is  $Y$  an ANR?

55. (*J. Heath, Kozłowski*) If  $r: S^n \rightarrow S^n/A$  is refinable and  $n > 3$ , must  $A$  be cellular?

[Answer is yes if  $n \leq 3$ .]

56. (*Edwards*) If  $f: S^3 \rightarrow S^2$  is a continuous surjection must there exist  $\Sigma^2$  (an embedded copy of  $S^2$  in  $S^3$ ) such that  $f|_{\Sigma^2}$  is a surjection?

[The analogous question for a map  $f: S^2 \rightarrow S^1$  has an affirmative solution.]

57. (*Boxer*) Do ARI maps preserve property K?

[A continuous surjection of compacta  $f: X \rightarrow Y$  is approximately right invertible (ARI) if there is a null sequence  $\{\varepsilon_n\}$  of positive numbers and a sequence of maps  $g_n: Y \rightarrow X$  such that  $d(fg_n, \text{id}_Y) < \varepsilon_n$  for each  $n$  ( $d = \text{sup-metric}$ ). A continuum  $X$  has property K if for each  $\varepsilon > 0$

there exists  $\delta > 0$  such that for each  $p \in X$  and each  $A \in C(X)$  with  $p \in A$ , if  $q \in X$  and  $\text{dist}(p, q) < \delta$ , then there exists  $B \in C(X)$  with  $q \in B$  and  $H(A, B) < \epsilon$ . ( $H$  = Hausdorff metric.  $C(X)$  = hyperspace of subcontinua of  $X$ .) If  $\{g_n\}$  is equicontinuous, the question has a positive answer. This does not represent new knowledge unless the next question has a negative answer for a continuum  $X$  with property  $K$ . The above question is a special case of a question in Nadler's book.]

58. (*Boxer*) If  $f: X \rightarrow Y$  is an ARI map with an equicontinuous sequence as in the above comments, is  $f$  an  $r$ -map?

### Homeomorphism Groups

59. (*Duda*) Is the map  $h: G(P) \rightarrow \mathbb{R}$ , ( $G(P)$  is the group of homeomorphisms of the pseudo-arc  $P$ ), defined by  $H(g) = \text{dist}(g, \text{identity})$ , a surjection onto  $[0, \text{diam } P]$ , or does the image at least contain a neighborhood (relative to  $[0, \text{diam } P]$ ) of 0?

60. (*Brechner*) Is the homeomorphism group of the pseudo-arc totally disconnected?

[Brechner and Anderson have proven an analogous result for the Menger universal curve. The homeomorphism group of the pseudo-arc contains no nondegenerate subcontinua, by a result of Lewis.]

61. (*Lewis*) Is the homeomorphism group of every hereditarily indecomposable continuum totally disconnected?

62. (*Lewis*) Must the homeomorphism group of a



homogeneous continuum either contain an arc or be totally disconnected?

63. (*Brechner*) If a homogeneous continuum  $X$  has a homeomorphism group which contains an arc (or admits non-trivial isotopies), must  $X$  admit a non-trivial flow?

64. (*Brechner*) Is the homeomorphism group of the pseudo-arc infinite dimensional?

65. (*Lewis*) Is the homeomorphism group of every non-degenerate homogeneous continuum infinite dimensional?

[Keesling has shown that if the homeomorphism group  $G(X)$  of a compact metric space  $X$  contains an arc, then  $G(X)$  is infinite dimensional.]

66. (*Lewis*) Is every connected subset of the space of continuous maps of the pseudo-arc into itself which contains a homeomorphism degenerate?

[The analogous result for the Menger universal curve is true.]

67. (*Lewis*) Is there a natural measure which can be put on the space  $M(P)$  of self-maps of the pseudo-arc? If so, what is the measure of the subspace  $\hat{H}(P)$  of maps which are homeomorphisms onto their image? Is it the same as the measure of  $M(P)$ ?

[ $\hat{H}(P)$  is a dense  $G_\delta$  in  $M(P)$ .]

68. (*Lewis*) Does the pseudo-circle have uncountably many orbits under the action of its homeomorphism group? What about other non-chainable continua all of whose non-degenerate proper subcontinua are pseudo-arcs?

[It can be shown that each orbit of such a continuum is dense, and that no such continuum has a  $G_\delta$  orbit.]

69. (*Wechsler*) If  $X$  and  $Y$  are homogeneous continua with isomorphic and homeomorphic homeomorphism groups, are  $X$  and  $Y$  homeomorphic?

[Whittaker has shown that compact manifolds with or without boundary are homeomorphic if and only if their homeomorphism groups are isomorphic. Rubin has shown that if  $X$  and  $Y$  are locally compact and strongly locally homogeneous, then they are homeomorphic if and only if their homomorphism groups are isomorphic. Sharma has shown that there are (non-metric) locally compact Galois spaces  $X$  and  $Y$  with isomorphic homeomorphism groups such that  $X$  and  $Y$  are not homeomorphic. van Mill has constructed non locally compact, connected subsets of the 2-sphere which are strongly locally homogeneous and have algebraically (but not topologically) isomorphic homeomorphism groups, but which are not themselves homeomorphic.]

70. (*Anzel*) Let  $G$  be the space of homeomorphisms of  $S^n$  and  $X$  the embeddings of  $S^{n-1}$  into  $S^n$ . Is there some condition analogous to 1 - ulc which will detect when the orbit in  $G$  of a given embedding is a  $G_\delta$ ? Is there also a way to distinguish non- $G_\delta$  orbits? Are there, in some reasonable sense, more non- $G_\delta$  orbits than  $G_\delta$  orbits?

71. (*Brechner*) If  $G$  is the collection of homeomorphisms of the pseudo-arc  $P$  which leave every composant invariant, is  $G$  dense in the full homeomorphism group of  $P$ ? of first category?

72. (*Brechner*) Do minimal normal subgroups of the groups of homeomorphisms characterize chainable continua?

[Also (*Brechner*): Find a nice characterization of normal subgroups of the homeomorphism group of the pseudo-arc.]

73. (*Jones*) What is the structure of the collection of homeomorphisms leaving a given point of the pseudo-arc fixed?

### Homogeneity

74. (*Jones*) Is every homogeneous, hereditarily indecomposable, nondegenerate continuum a pseudo-arc?

[Rogers has shown that it must be tree-like.]

75. (*Jones*) Is each nondegenerate, homogeneous, non-separating plane continuum a pseudo-arc?

[Jones and Hagopian have shown that it must be hereditarily indecomposable. Rogers has shown that it must be tree-like.]

76. (*Fitzpatrick*, UHPB 88) Is every homogeneous continuum bihomogeneous?

[ $X$  is bihomogeneous if for every  $x_0, x_1 \in X$  there exists a homeomorphism  $h: X \rightarrow X$  with  $h(x_i) = x_{1-i}$ .]

77. (*Jones*) What effect does hereditary equivalence have on homogeneity in continua?

78. (*Hagopian*) If a homogeneous continuum  $X$  contains an arc must it contain a solenoid or a simple closed curve? What if  $X$  contains no simple triod?

79. (*Rogers*) Is each acyclic, homogeneous, one-dimensional continuum tree-like? hereditarily indecomposable?

80. (*K. Kuperberg*) Does there exist a homogeneous, arcwise-connected continuum which is not locally connected?

81. (*Minc*) Is the simple closed curve the only nondegenerate, homogeneous, hereditarily decomposable continuum?

82. (*Gordh*) Is every hereditarily unicoherent, homogeneous,  $T_2$  continuum indecomposable?

[Jones has shown that the metric version of this question has an affirmative answer.]

83. (*Ungar*) Is every finite-dimensional, homeotopically homogeneous continuum a manifold?

[X is homeotopically homogeneous if for each  $x, y \in X$  there is a homeomorphism  $h: (X, x) \rightarrow (X, y)$  and an isotopy connecting  $h$  to the identity.]

84. (*Burgess*) Is every  $n$ -homogeneous continuum  $(n + 1)$ -homogeneous for each  $n \geq 2$ ?

[A continuum  $X$  is  $n$ -homogeneous if for each pair of collections  $\{x_1, x_2, \dots, x_n\}$  and  $\{y_1, y_2, \dots, y_n\}$  of  $n$  distinct points of  $X$  there is a homeomorphism  $h: X \rightarrow X$  with  $h(\{x_1, x_2, \dots, x_n\}) = \{y_1, y_2, \dots, y_n\}$ . Ungar has shown that if  $X$  is  $n$ -homogeneous and  $X \neq S^1$  then  $h$  can also be chosen such that  $h(x_i) = y_i$  for each  $1 \leq i \leq n$ . Kennedy has shown that if  $X$  is  $n$ -homogeneous ( $n \geq 2$ ) and admits a non-identity stable homeomorphism then  $X$  is  $m$ -homogeneous for each positive integer  $m$  (and in fact countable dense homogeneous and

representable). Ungar has shown that if  $X$  is  $n$ -homogeneous ( $n \geq 2$ ) then  $X$  is locally connected.]

85. (*Kennedy*) Does every non-degenerate homogeneous continuum admit a non-identity stable homeomorphism?

86. (*Bing*) Is every homogeneous tree-like continuum hereditarily indecomposable?

[Jones and Hagopian have shown that in the plane the answer is yes. Jones has shown that such a continuum must be indecomposable. Hagopian has shown that it cannot contain an arc. Each of the following variants has been asked by various persons at various times. Is each such non-degenerate continuum a pseudo-arc? weakly chainable? hereditarily equivalent? of span zero? a continuum with the fixed point property?]

87. (*Rogers*) Does every indecomposable, homogeneous continuum have dimension at most one?

88. (*Rogers*) Is each aposyndetic, non-locally-connected, one-dimensional, homogeneous continuum an inverse limit of Menger curves and continuous maps? Menger curves and fibrations? Menger curves and covering maps? Is each a Cantor set bundle over the Menger curve?

89. (*Minc*) Can each aposyndetic, non-locally connected, one-dimensional homogeneous continuum be mapped onto a solenoid?

[*Rogers*: Can such a continuum be retracted onto a non-trivial solenoid? Does each such continuum contain an arc?]

90. (*Rogers*) Is each pointed-1-movable, aposyndetic, homogeneous one-dimensional continuum locally connected?

91. (*Rogers*) Must each cyclic, indecomposable, homogeneous, one-dimensional continuum either be a solenoid or admit a continuous decomposition into tree-like, homogeneous continua with quotient space a solenoid?

92. (*Rogers*) Is every decomposable, homogeneous continuum of dimension greater than one aposyndetic?

93. (*Rogers*) Can the Jones Decomposition Theorem be strengthened to give decomposition elements which are hereditarily indecomposable? Can such a decomposition raise dimension? lower dimension?

94. Let  $X$  be a non-degenerate, homogeneous, contractible continuum. Is  $X$  an AR? Is  $X$  homeomorphic to the Hilbert cube?

95. (*Patkowska*) What are the homogeneous Peano continua in  $E^3$ ?

96. (*Patkowska*) Does there exist a 2-homogeneous continuum  $X = X_1 \times X_2$ , where  $X_1$  and  $X_2$  are non-degenerate, which is not either a manifold or an infinite product of manifolds?

97. (*Bellamy*) Is the following statement false?  
Statement: Suppose  $X$  is a homogeneous compact connected  $T_2$  space. Then for every open cover  $U$  of  $X$  there is an open cover  $V$  of  $X$  such that whenever  $x$  and  $y$  belong to the same

element of  $V$  there is a homeomorphism  $h: X \rightarrow X$  such that  $h(x) = y$  and such that for every  $p \in X$ ,  $p$  and  $h(p)$  belong to the same element of  $U$ .

98. (*Bellamy*) If  $X$  is an arcwise connected homogeneous continuum other than a simple closed curve, must each pair of points be the vertices of a  $\theta$ -curve in  $X$ ?

[Bellamy and Lum have shown that each pair of points of  $X$  must lie on a simple closed curve.]

99. (*Bellamy*) Does each finite subset of a nondegenerate arcwise connected homogeneous continuum lie on a simple closed curve?

100. (*Bellamy*) Does each nondegenerate arcwise connected homogeneous continuum other than the simple closed curve contain simple closed curves of arbitrarily small diameter?

101. (*Wilson*) Does there exist a uniquely arcwise connected homogeneous compact  $T_2$  continuum, with an arc being defined either as a homeomorph of  $[0,1]$  or as a compact  $T_2$  continuum with exactly two nonseparating points?

[By a result of Bellamy and Lum, such a continuum cannot be metric.]

102. (*Lewis*) Does there exist a homogeneous one-dimensional continuum with no non-degenerate chainable subcontinua?

[If there exists a nondegenerate, homogeneous, hereditarily indecomposable continuum other than the pseudo-arc, the answer is yes.]

103. (*Bennett*) Is each open subset of a countable dense homogeneous continuum itself countable dense homogeneous?

[M is countable dense homogeneous if for each two countable dense subsets S and T of M there is a homeomorphism  $h: M \rightarrow M$  with  $h(S) = T$ .]

104. (*Fearnley*) Is every continuum a continuous image of a homogeneous continuum? In particular, is the spiral around a triod such an image?

105. (*J. Charatonik*) Is the Sierpinski curve homogeneous with respect to open surjections?

### Hyperspaces

In each of the following, X is a metric continuum, and  $C(X)$  (resp.  $2^X$ ) is the hyperspace of subcontinua (resp. closed subsets) of X with the Hausdorff metric.

106. (*Rogers*) If  $\dim X > 1$ , is  $\dim C(X) = \infty$ ? What if X is indecomposable?

[Rogers raised the question and conjectured at the U.S.L. Mathematics Conference in 1971 that the answer is yes. The answer is known to be yes if any of the following are added to the hypothesis: (1) X is locally connected; (2) X contains the product of two nondegenerate continua; (3)  $\dim X > 2$ ; or (4) X is hereditarily indecomposable.]

107. (*Rogers*) If  $\dim X = 1$  and X is planar and atriodic, is  $\dim C(X) = 2$ ? Is  $C(X)$  embeddable in  $\mathbb{R}^3$ ?

[The answer is yes if X is either hereditarily indecomposable or locally connected.]



108. (*Rogers*) If  $\dim X = 1$  and  $X$  is hereditarily decomposable and atriodic, is  $\dim C(X) = 2$ ?

109. (*Rogers*) If  $X$  is tree-like, does  $C(X)$  have the fixed point property?

110. (*Nadler*) When does  $2^X$  have the fixed point property?

111. (*Dilks*) Is  $C(X)$  or  $2^X$  locally contractible at the point  $X$ ?

112. (*Rogers*) Are any of the following Whitney properties: (a)  $\delta$ -connected; (b) weakly chainable; or (c) pointed-one-movable?

[Krasinkiewicz and Nadler have asked which of the following are Whitney properties: acyclic, ANR, AR, contractibility, Hilbert cube, homogeneity,  $\lambda$ -connected,  $\text{Sh}(X) < \text{Sh}(Y)$ , and weakly chainable. W. Charatonik has recently shown that homogeneity is not a Whitney property.]

113. (*Dilks and Rogers*) Let  $X$  be finite-dimensional and have the cone = hyperspace property. Must  $X$  have property  $K$ ? belong to class  $W$ ? be Whitney stable?

### Inverse Limits

114. (*Young*) Is there for each  $k \geq 1$  an atriodic tree-like continuum which is level  $(k + 1)$  but not level  $k$  (equivalently: Burgess'  $(k + 1)$ -junctioned but not  $k$ -junctioned). The equivalent question for  $(k + 1)$ -branched but not  $k$ -branched. Find a useful way to characterize level  $n$ .

[A tree-like continuum  $M$  is level  $n$  if for every  $\varepsilon > 0$  there exists an  $\varepsilon$ -map of  $M$  onto a tree with  $n$  points of order greater than two.]

115. (*Young*) Is there a continuum which is 4-od like, not T-like, and every nondegenerate proper subcontinuum of which is an arc?

116. Under what conditions is the inverse limit of dendroids a dendroid?

[A dendroid is an arcwise connected, hereditarily unicoherent continuum.]

117. (*Bellamy*) Define  $f_a: [0,1] \rightarrow [0,1]$  by  $f(t) = at(1-t)$  for  $0 \leq a \leq 4$ . Is there a relationship between the existence of periodic points of  $f_a$  of various periods and the topological nature of the inverse limit continuum obtained by using  $f_a$  as each one-step bonding map? In particular is the inverse limit continuum indecomposable if and only if  $f_a$  has a point of period 3?

### Mapping Properties

118. (*W. Kuperberg*, UHPB 31) Is it true that the pseudo-arc is not pseudo-contractible?

[A continuum  $X$  is pseudo-contractible if there exists a continuum  $Y$ , points  $a, b \in Y$  and a map  $h: X \times Y \rightarrow X$  such that  $h_a: X \times \{a\} \rightarrow X$  is a homeomorphism and  $h_b: X \times \{b\} \rightarrow X$  is a constant map. Also (*W. Kuperberg*, UHPB 29): Does there exist a one-dimensional continuum which is pseudo-contractible but not contractible?]

119. (Maćkowiak) Does there exist a chainable continuum  $X$  such that if  $H$  and  $K$  are subcontinua of  $X$  then the only maps between  $H$  and  $K$  are the identity or constants?

[Maćkowiak has constructed a chainable continuum which admits only the identity or constants as self maps.]

120. (Lewis) Is every subcontinuum of a weakly chainable, atriodic, tree-like continuum weakly chainable?

121. (Lewis) If  $P$  is the pseudo-arc and  $X$  is a nondegenerate continuum, is  $P \times X$  Galois if and only if  $X$  is isotopy Galois?

[ $X$  is Galois if for each  $x \in X$  and open  $U$  containing  $x$  there exists a homeomorphism  $h: X \rightarrow X$  with  $h(x) \neq x$  and  $h(z) = z$  for each  $z \in U$ . If in addition  $h$  can be chosen isotopic to the identity, each level of the isotopy satisfying  $h(z) = z$  for each  $z \notin U$ , then  $X$  is isotopy Galois. The parallel question for the Menger curve has a positive answer.]

122. (Lewis) If  $h$  is a homeomorphism of  $\prod_{\alpha \in A} P_\alpha$ , where each  $P_\alpha$  is a pseudo-arc, is  $h$  necessarily of the form  $h = \prod_{\alpha \in A} h_{s(\alpha)}$ , where  $s$  is a permutation of  $A$  and  $h_{s(\alpha)}$  is a homeomorphism of  $P_\alpha$  onto  $P_{s(\alpha)}$ ?

[Bellamy and Lysko have given a positive answer when  $A$  contains at most two elements. Cauty has shown the parallel question has a positive answer for any product of one-dimensional continua each open subset of which contains a simple closed curve (e.g. Menger curves or Sierpinski curves).]

123. (*Eberhart*) If  $X$  is a locally compact, metric space with every proper subcontinuum of  $X$  hereditarily indecomposable, and  $f$  is a local homeomorphism on  $X$ , is  $f$  a homeomorphism on proper subcontinua of  $X$ ?

124. (*Bellamy*) Conjecture: Let  $X$  be a non-degenerate metric continuum,  $p \in X$ . Then there exist mappings  $H: C \rightarrow C(X)$ , ( $C$  = Cantor set,  $C(X)$  = hyperspace of subcontinua of  $X$ ) and  $h: C \rightarrow X$  such that  $H$  and  $h$  are embeddings and for each  $x \in C$ ,  $H(x)$  is irreducible from  $p$  to  $h(x)$  and if  $x, y \in C$ ,  $x < y$  (in ordering as a subset of  $[0,1]$ ), then  $H(x) \subsetneq H(y)$ .

125. (*Mine*) Suppose  $X$  is a plane continuum such that for each  $x, y \in X$  there is a weakly chainable subcontinuum of  $X$  containing both  $x$  and  $y$ . Is  $X$  weakly chainable?

[Special case: Suppose  $X$  is arcwise connected. Answer may be no if  $X$  is non-planar.]

126. (*Young*) Suppose that  $f$  is a light map of a tree  $T_1$  onto a tree  $T_2$  with the following property: Given light maps  $g, h$  from the unit interval  $I$  onto  $T_1$ , there exist maps  $\alpha, \beta: I \rightarrow I$  such that  $fg\alpha = fh\beta$ . Does  $f$  factor through an arc? What if all maps are piecewise linear?

127. (*Oversteegen*) Suppose  $X$  is a weakly chainable, tree-like continuum. Do there exist inverse sequences  $\lim_{\leftarrow} (I_n, g_n) \approx P$  ( $P$  = pseudo-arc,  $I_n$  = unit interval),  $\lim_{\leftarrow} (T_n, f_n) \approx X$  (each  $T_n$  a tree), and maps  $h_n: I_n \rightarrow T_n$  such that  $h = \lim_{\leftarrow} h_n = P \rightarrow X$ ?

[Mioduszewski has shown that the answer is yes if  $X$  is arc-like.]

128. (*Oversteegen*) Suppose  $X$  is a continuum such that for each  $x \in X$  there exists a neighborhood  $U_x$  of  $x$  such that  $U_x \approx (0,1) \times A$  ( $A =$  compact, zero-dimensional set). Is  $X$  not tree-like?

129. (*Bellamy*) Suppose  $X$  is a non-pointed-one-movable continuum. Is there a non-pointed-one-movable continuum  $M(X)$  which is either circle-like or figure-eight-like onto which  $X$  can be mapped?

130. (*Krasinkiewicz*) Is there a finite-to-one map of an hereditarily indecomposable continuum onto an hereditarily decomposable continuum?

131. (*Bellamy*) For countable non-limit ordinals  $\alpha$ , what are the continuous images of  $C(\alpha)$ , the cone over  $\alpha$ ? For  $\alpha \geq \omega^2 + 1$ , what are the continuous pre-images of  $C(\alpha)$ ?

132. (*Bellamy*) Is every continuous image of the cone over the Cantor set  $g$ -contractible?

[A continuum is  $g$ -contractible if and only if it admits a null-homotopic self surjection.]

133. (*Bellamy*) If an hereditarily indecomposable continuum admits an essential map onto a circle, does it admit a map onto a pseudo-circle?

134. (*Bellamy*) Does every finite dimensional, hereditarily indecomposable continuum embed into a finite product of pseudo-arcs?

135. (*Bellamy*) Does every one dimensional hereditarily indecomposable continuum embed in a product of three (or maybe even two) pseudo-arcs?

136. (*Bellamy*) Does every tree-like hereditarily indecomposable continuum embed into a product of two (or three) pseudo-arcs? Does every planar hereditarily indecomposable continuum embed in a product of two pseudo-arcs?

137. (*Bellamy*) Is the pseudo-circle a retract of every one-dimensional hereditarily indecomposable continuum containing it?

### Problems in the Plane

138. (*Lewis*) Does every hereditarily indecomposable plane continuum have  $c = 2^{\omega_0}$  distinct embeddings in  $E^2$ ? Does each such continuum have, for each integer  $n > 1$ , an embedding with exactly  $n$  accessible composants? Does every such continuum have an embedding with no two accessible points in the same composant?

139. (*Burgess*) Which continua in  $E^2$  have the property that all of their embeddings in  $E^2$  are equivalent?

140. (*Nadler and Quinn*) If  $p$  is a point of the chainable continuum  $M$ , is there an embedding of  $M$  in  $E^2$  which makes  $p$  accessible?

141. (*Mayer*) Are there uncountably many inequivalent embeddings of every chainable indecomposable continuum in  $E^2$ ?

142. (*Mayer*) Can every chainable indecomposable continuum be embedded in  $E^2$  non-principally (i.e. without a simple dense canal)?

[This is known for such continua with at least one endpoint.]

143. (*Brechner and Mayer*) Does there exist a non-separating plane continuum such that every embedding of it in  $E^2$  has a simple dense canal?

144. (*Anceł*) Is every embedding of a Peano continuum in  $R^2$  micro-unknotted? Is the standard inclusion  $S^3 \rightarrow S^4$  micro-unknotted?

[Suppose  $M$  and  $N$  are compact, metric spaces,  $G$  is the homeomorphism group of  $N$ , and  $X$  is the space of embeddings of  $M$  in  $N$ . An embedding  $e: M \rightarrow N$  is micro-unknotted if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $h \in G$  and  $\text{dist}_X(e, h \circ e) < \delta$ , then there exists  $h' \in G$  with  $\text{dist}_G(1_N, h') < \varepsilon$  and  $h' \circ e = h \circ e$ .  $e: M \rightarrow N$  is micro-unknotted  $\iff G$  acts micro-transitively on the orbit  $G \circ e \iff G \circ e$  is  $G_\delta$  in  $X$  (Effros' theorem).]

145. (*Jones*) What characterizes dendroids that are embeddable in  $E^2$ ? What characterizes dendroids that are contractible?

146. (*Anceł*) Is there a recognizable family of non-separating plane continua such that every non-separating plane continuum is a retract of a member of this family?

147. (*Bellamy*) When is the inverse image  $S$  of an indecomposable plane continuum  $X$  under a complex power map ( $f(z) = z^n$  for some  $n$ ) itself an indecomposable continuum? In particular, if  $0$  lies in an inaccessible composant of  $X$ , is  $S$  indecomposable?

148. Suppose  $M$  is a non-degenerate connected subset of  $E^2$ , such that the complement of each point in  $M$  is connected but the complement of each pair of points in  $M$  is disconnected. Can  $E^2 - M$  be arcwise connected?

### Set Function $T$

Let  $S$  be a compact Hausdorff space, and let  $A$  be a subset of  $S$ .  $T(A)$  is the set of points which have no continuum neighborhood missing  $A$ .  $K(A)$  is the intersection of all continuum neighborhoods of  $A$ . The following problems are unsolved for compact Hausdorff continua, with the possible exception of number 157. Except for number 158, they are unsolved for compact metric continua.

The phrase " $T$  is continuous for  $S$ " means that  $T$  is continuous considered as a function from the hyperspace of closed subsets of  $S$  to itself; similarly for  $K$ . " $S$  is  $T$ -additive" means that for closed sets  $A, B \subseteq S$ ,  $T(A \cup B) = T(A) \cup T(B)$ . All questions in this section were posed by Bellamy unless indicated otherwise.

149. If  $T$  is continuous for  $S$ , is  $K$  also continuous for  $S$ ?

150. If  $T$  is continuous for  $S$  and  $S$  is decomposable, is it true that for each  $p \in S$ ,  $\text{Int}(T(p)) = \emptyset$ ?



151. If  $T$  is continuous for  $S$ , is  $S$   $T$ -additive?

[Bellamy has offered a prize for the solution of this question--one bushel of extra fancy Stayman Winesap apples, delivered in season.]

152. If  $S/T$  denotes the finest decomposition space of  $S$  which shrinks each  $T(p)$  to a point, is  $S/T$  locally connected?

[This is not difficult to show if  $S$  is also  $T$  additive.]

153. (*Jones*) If  $X$  is indecomposable and  $W$  is a subcontinuum of  $X \times X$  with non-empty interior, is  $T(W) = X \times X$ ?

154. (*Cook*) If  $X$  is atriodic (or contains no uncountable collection of pairwise disjoint triods) and  $X$  has no continuum cut point, does this imply that there is a continuum  $W \subset X$  such that  $\text{Int}(W) \neq \emptyset$  and  $T(W) \neq X$ ?

155. If  $T$  is continuous for  $S$  and  $f: S \rightarrow Z$  is a continuous and monotone surjection, is  $T$  continuous for  $Z$  also?

156. If  $X$  is one-dimensional and homogeneous is  $T$  continuous for  $S$ ?

157. Call a continuum  $S$  *strictly point  $T$  asymmetric* if for  $p \neq q$  and  $p \in T(q)$  we have  $q \notin T(p)$ . In dendroids, does this property imply smoothness?

158. (*H. Davis and Doyle*) If  $S$  is almost connected im kleinen, is  $S$  connected im kleinen at some point?

[Almost connectedness im kleinen can be expressed in terms of the set function  $T$  as follows:  $S$  is almost

connected im kleinen at  $p \in S$  if and only if for each closed  $A$  for which  $p \in \text{Int}(T(A))$  we have  $p \in \text{Int}(A)$ . This question is known to be true for the metric case.]

159. Suppose the restriction of  $T$  to the hyperspace of subcontinua of  $S$  is continuous. Does this imply that  $T$  is continuous for  $S$ ?

[This is true if  $T$  is the identity on subcontinua.]

160. Do open maps preserve  $T$ -additivity?  $T$ -symmetry?

[ $S$  is  $T$ -symmetric if and only if for all closed sets  $A$  and  $B$  in  $S$ , if  $A \cap T(B) = \emptyset$  then  $B \cap T(A) = \emptyset$ .]

### Span

161. (*Lelek, Cook*, UHPB 81) Is each continuum of span zero chainable?

162. (*Duda*) To what extent does span zero parallel chainability, i.e.

(i) Is the open image of a continuum of span zero a continuum of span zero?

(ii) (*Lelek*, UHPB 84) Is the confluent image of a chainable continuum chainable?

(iii) (*Lelek, Cook*, UHPB 86) Do confluent maps of continua preserve span zero?

[Also (*Lelek*, UHPB 85): If  $f$  is a confluent mapping of an acyclic (or tree-like or arc-like) continuum  $X$  onto a continuum  $Y$ , is  $f \times f$  confluent? An affirmative solution to (ii) would show that the classification of homogeneous plane continua is complete. McLean has shown that the

confluent image of a tree-like continuum is tree-like, and Rosenholtz that the open image of a chainable continuum is chainable.]

163. (*Cook*, UHPB 92) If  $M$  is a continuum with positive span such that each of its proper subcontinua has span zero, does every nondegenerate, monotone, continuous image of  $M$  have positive span?

164. (*Cook*, UHPB 173) Do there exist, in the plane, two simple closed curves  $J$  and  $K$  such that  $K$  is in the bounded complementary domain of  $J$ , and the span of  $K$  is greater than the span of  $J$ ?