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PERVIN NEARNESS SPACES

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Introduction

The collection of all finite open covers of a topological space generates the Pervin quasi-uniform structure for that space. All covers refined by some finite open cover forms a nearness structure for a symmetric topological space. This nearness structure will be called the Pervin nearness structure.

The Pervin nearness structure plays an interesting role in the family of all compatible nearness structures on a space. It is the smallest totally bounded structure, the smallest contigual structure, and the largest ultrafilter generated structure.

Herrlich, in [10], has shown that the completion of the contigual reflection of a T_1 topological nearness space is the Wallman compactification. But the contigual reflection of a topological nearness structure is the Pervin nearness structure. Hence, for T_1 spaces, the Pervin nearness structure is induced by the Wallman compactification.

A prime extension is one for which each trace filter is a prime open filter. Since the Pervin nearness structure is ultrafilter generated it follows that the Wallman compactification or the Stone-Čech compactification, if the space is normal, are prime extensions.

For a T_1 space it is shown that the trace filters of the Wallman compactification are the minimal prime open filters. Moreover, we can construct the Wallman compactification for a T_1 space, or the Stone-Čech compactification for a normal space, using the strict extension topology on the family of all minimal prime open filters.

For T_1 spaces, concrete nearness structures ξ on X are induced by a strict extension Y. There is a one-to-one correspondence between the points of Y and the ξ-clusters. Balanced near collections are introduced and it is shown that there is a one-to-one correspondence between the nonempty closed subsets of Y and the balanced near collections on X. This correspondence applied to the Pervin nearness structure yields a correspondence between the nonempty closed subsets of the Wallman compactification and the balanced closed filters on X. Using this result, certain closed sets in βN are characterized in terms of certain filters on N.

1. Preliminaries

We will assume that the reader is basically familiar with the concept of a nearness space as defined by Herrlich in [9] and [10].

Definition 1.1. Let X be a set and μ a collection of covers of X, called uniform covers. Then (X,μ) is a nearness space provided:

(N1) $\mathcal{A} \in \mu$ and $\mathcal{A}$ refines $\mathcal{B}$ implies $\mathcal{B} \in \mu$.

(N2) $\{X\} \in \mu$ and $\phi \notin \mu$.

(N3) If $\mathcal{A} \in \mu$ and $\mathcal{B} \in \mu$ then $\mathcal{A} \wedge \mathcal{B} = \{A \cap B\colon A \in \mathcal{A}$ and $B \in \mathcal{B}\} \in \mu$.

(N4) $\mathcal{A} \in \mu$ implies $\{\text{int}(A): A \in \mathcal{A}\} \in \mu$. ($\text{int}(A) = \{x: \{X-\{x\},A\} \in \mu\}$).

For a given nearness space (X,μ) the collection of sets that are "near" is given by $\xi = \{\mathcal{A} \subset \mathcal{P}(X): \{X-A: A \in \mathcal{A}\} \notin \mu\}$. The micromeric collections are given by $\mathcal{A} \in \gamma$ if and only if $\{B \subset X: A \cap B \neq \phi$ for each $A \in \mathcal{A}\} \in \xi$. The closure operator generated by a nearness space is given by $\text{cl}_\xi A = \{x: \{\{x\},A\} \in \xi\}$. If we are primarily using these "near" collections we will denote the nearness space by (X,ξ). The underlying topology of a nearness space is always symmetric; that is, $x \in \overline{\{y\}}$ implies $y \in \overline{\{x\}}$.

Definition 1.2. Let (X,ξ) be a nearness space. The nearness space is called:

(1) topological provided $\mathcal{A} \in \xi$ implies $\cap\overline{\mathcal{A}} \neq \phi$.

(2) complete provided each ξ-cluster is fixed; that is, $\cap\overline{\mathcal{A}} \neq \phi$ for each maximal element $\mathcal{A}$ in ξ.

(3) concrete provided each near collection is contained in some ξ-cluster.

(4) contigual provided $\mathcal{A} \notin \xi$ implies there exists a finite $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{B} \notin \xi$.

(5) totally bounded provided $\mathcal{A} \notin \xi$ implies there exists a finite $\mathcal{B} \subset \mathcal{A}$ such that $\cap\mathcal{B} = \phi$.

2. Closed Filters

The following notation will be used in this paper.

Definition 2.1. Let $\mathcal{F}$ be a closed filter in a topological space (X,t).

(1) $\mathcal{G}(\mathcal{F}) = \{A \subset X: \overline{A} \in \mathcal{F}\}$.

(2) $\mathcal{O}(\mathcal{F}) = \{0 \in t$: there exists $F \in \mathcal{F}$ with $F \subset 0\}$.

(3) $\sec(\mathcal{F}) = \{A \subset X$: A is closed and $A \cap F \neq \phi$ for each $F \in \mathcal{F}\}$.

(4) $\sec^2(\mathcal{F}) = \sec(\sec(\mathcal{F}))$.

(5) If $\overline{\mathcal{A}}$ has the finite intersection property then $\mathcal{F}(\overline{\mathcal{A}})$ will denote the closed filter generated by $\overline{\mathcal{A}}$.

(6) $\mathcal{F}$ is called balanced provided $\mathcal{F} = \cap\{\mathcal{M}$: $\mathcal{M}$ is a closed ultrafilter containing $\mathcal{F}\}$.

(7) $\mathcal{F}$ is called rigid provided $\mathcal{F} = \cap\{\mathcal{M}$: $\mathcal{M}$ is a fixed closed ultrafilter containing $\mathcal{F}\}$.

(8) $\mathcal{F}$ is called nontrivial provided $\mathcal{F} \neq \{X\}$.

(9) $\mathcal{F}$ is called a sparce closed filter if for each nontrivial closed filter $\mathcal{H}$ there exists a closed ultrafilter $\mathcal{G}$ with $\mathcal{H} \not\subset \mathcal{G}$ and $\mathcal{F} \not\subset \mathcal{G}$.

(10) A collection of closed filters $\{\mathcal{F}_\alpha: \alpha \in I\}$ is called uniformly sparce if for each nontrivial closed filter $\mathcal{H}$ there exists a closed ultrafilter $\mathcal{G}$ such that $\mathcal{H} \not\subset \mathcal{G}$ and $\mathcal{F}_\alpha \not\subset \mathcal{G}$ for each $\alpha \in I$.

Lemma 2.1. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be closed filters on a topological space. Then $\mathcal{F}_1 \subset \mathcal{F}_2$ if and only if $\mathcal{G}(\mathcal{F}_1) \subset \mathcal{G}(\mathcal{F}_2)$.*

Lemma 2.2. *Let $\mathcal{F}$ be a closed filter on a topological space.*

(1) *$\mathcal{F} = \cap\{\mathcal{M}$: $\mathcal{M}$ is a closed ultrafilter containing $\mathcal{F}\}$ if and only if $\mathcal{G}(\mathcal{F}) = \cap\{\mathcal{G}(\mathcal{M})$: $\mathcal{M}$ is a closed ultrafilter containing $\mathcal{F}\}$.*

(2) $\sec(\mathcal{F}) = \cup\{\mathcal{M}$: *$\mathcal{M}$ is a closed ultrafilter containing* $\mathcal{F}\}$.

(3) $\sec^2(\mathcal{F}) = \cap\{\mathcal{M}$: *$\mathcal{M}$ is a closed ultrafilter containing $\mathcal{F}$*$\}$.

(4) $\sec^2(\mathcal{F})$ *is a closed filter containing $\mathcal{F}$.*

(5) *$\mathcal{F}$ is balanced if and only if* $\mathcal{F} = \sec^2(\mathcal{F})$.

(6) *$\mathcal{F}$ is rigid if and only if there exists a nonempty closed set* A *such that* $\mathcal{F} = \mathcal{F}(\{A\})$.

Lemma 2.3. *Let* (X,t) *be a topological space.*

(1) *If* X *is* T_1 *and* $x \in X$ *then* $\mathcal{O}_x$, *the open neighborhood filter of* x, *is a minimal prime open filter.*

(2) *If $\mathcal{F}$ is a closed ultrafilter then* $\mathcal{O}(\mathcal{F}) = \{0 \in t: X-0 \notin \mathcal{F}\}$.

(3) *$\mathcal{O}$ is a minimal prime open filter if and only if there exists a closed ultrafilter $\mathcal{F}$ such that* $\mathcal{O} = \mathcal{O}(\mathcal{F})$.

Proof. To show (3), let $\mathcal{O}$ be a minimal prime open filter and set $\mathcal{F} = \{F$: F is closed in X and $X-F \notin \mathcal{O}\}$. Then $\mathcal{F}$ is a closed ultrafilter and $\mathcal{O} = \mathcal{O}(\mathcal{F})$. The remainder of the proof follows in a natural way.

Lemma 2.4. *Let* (X,t) *be a* T_1 *topological space and $\mathcal{F}$ a closed filter on* X. *The following statements are equivalent.*

(1) *$\mathcal{F}$ is a sparce closed filter.*

(2) *For each nonempty open set* 0 *there exists* $F \in \mathcal{F}$ *such that* $0 \not\subset F$.

3. Pervin Nearness Structure

For a topological space (X,t) the collection $\{S(0): 0 \in t\}$ where $S(0) = (0 \times 0) \cup ((X-0) \times X)$, is a subbase for a compatible quasi-uniform structure on X called the Pervin quasi-uniform structure.

The Pervin quasi-uniform structure can also be generated as a covering quasi-uniform structure using the family of all finite open covers [7]. It is shown in [5] that the collection of all covers refined by a finite open cover is a compatible nearness structure. It is natural to call this the Pervin nearness structure.

Definition 3.1. Let (X,t) be a symmetric topological space. Let μ_p be the family of all covers of X that are refined by a finite open cover. μ_p is called the Pervin nearness structure on X.

For the purposes of this paper it is more convenient to work with ξ_p, the corresponding family of near collections.

Theorem 3.1. *Let* (X,t) *be a symmetric topological space. Then:* $\xi_p = \{\mathcal{A} \subset \mathcal{P}(X) : \overline{\mathcal{A}}$ *has the finite intersection property*$\}$.

From [3], we have the following result.

Theorem 3.2. *Let* (X,t) *be a symmetric topological space. Then:*

(1) X *is compact if and only if* ξ_p *is ultrafilter complete.*

(2) X *is* H-*closed if and only if* X *is Hausdorff and* ξ_p *is open ultrafilter complete.*

(3) X *is countably compact if and only if* $\overline{\mathcal{F}}$ *has the countable intersection property for each filter* $\mathcal{F}$ *in* ξ_p.

(4) X *is Lindelöf if and only if every filter in* ξ_p *with the countable intersection property clusters.*

Every contigual nearness structure is totally bounded; and it is easy to see that ξ_p is contigual. In fact, the smallest compatible totally bounded nearness structure on a symmetric topological space is ξ_p and thus equals the smallest compatible contigual nearness structure on the space.

Theorem 3.3. *Let* (X,t) *be a symmetric topological space. Then:*

(1) ξ_p *is contigual.*

(2) ξ_p *is the smallest compatible contigual nearness structure on* X.

(3) $\xi_p = \cap\{\xi: \xi$ *is a compatible contigual nearness structure on* X$\}$.

(4) $\xi_p = \cap\{\xi: \xi$ *is a compatible totally bounded nearness structure on* X$\}$.

(5) ξ_p *is the smallest compatible totally bounded nearness structure on* X.

(6) *The Pervin nearness structure is contained in each compatible totally bounded nearness structure on* X.

Definition 3.2. Let (X,t) be a symmetric topological space and $\mathcal{S}$ a collection of free ultrafilters on X. Set $\xi(\mathcal{S}) = \{\mathcal{A} \subset \mathcal{P}(X): \cap\overline{\mathcal{A}} \neq \phi$ or there exists $\mathcal{F} \in \mathcal{S}$ such that $\overline{\mathcal{A}} \subset \mathcal{F}\}$. A nearness structure ξ is called ultrafilter generated provided there exists $\mathcal{S}$ such that $\xi = \xi(\mathcal{S})$.

It is shown, in [8], that $\xi(\mathcal{S})$ is a compatible nearness structure on (X,t).

Theorem 3.4. *Let* (X,t) *be a symmetric topological space. Then the Pervin nearness structure is the largest compatible ultrafilter generated nearness structure on* (X,t).

Thus, for a given symmetric topological space, the Pervin nearness structure is the largest ultrafilter generated nearness structure and the smallest totally bounded nearness structure compatible with (X,t). Symbolically:

$$\xi_p = \text{Totally Bounded} \cap \text{Ultrafilter Generated}$$

Corollary 3.5. *Let* (X,t) *be a symmetric topological space. Let* ξ_1 *be any compatible totally bounded nearness structure and* ξ_2 *any compatible ultrafilter generated nearness structure. Then* $\xi_2 \subset \xi_1$. *That is, each compatible totally bounded nearness structure contains every compatible ultrafilter generated nearness structure.*

Since every contigual nearness structure is concrete we have the following corollary.

Corollary 3.6. *Every Pervin nearness structure is concrete.*

Theorem 3.7. *Let* (X,t) *be a symmetric topological space. Then:*

(1) $\xi_p = \{\mathcal{A} \subset \mathcal{P}(X)$: *$\mathcal{A} \subset \mathcal{G}(\mathcal{F})$ for some closed ultrafilter $\mathcal{F}$*$\}$.

(2) *$\mathcal{A}$ is a ξ_p-cluster if and only if there exists a closed ultrafilter $\mathcal{F}$ such that $\mathcal{A} = \mathcal{G}(\mathcal{F})$.*

(3) $\gamma_p = \{\mathcal{A} \subset \mathcal{P}(X)$: *there exists a minimal prime open filter $\mathcal{O}$ that corefines $\mathcal{A}$*$\}$. *(Note: $\mathcal{O}$ is said to corefine $\mathcal{A}$ if for each* $0 \in \mathcal{O}$ *there exists an* $A \in \mathcal{A}$ *such that* $A \subset 0$.)

4. Extensions

An extension Y of a space X is a space in which X is densely embedded. Unless otherwise noted, we will assume for notational convenience that $X \subset Y$. It is well known that for any extension Y of X there exists an equivalent extension Y' with $X \subset Y'$.

If Y is an extension of X then $\xi = \{\mathcal{A} \subset \mathcal{P}(X)$: $\cap \mathrm{cl}_Y \mathcal{A} \neq \phi\}$ is called the nearness structure on X induced by Y.

Let (Y,t) be a topological space and $\overline{X} = Y$. For each $y \in Y$, set $\mathcal{O}_y = \{0 \cap X: y \in 0 \in t\}$. Then $\{\mathcal{O}_y: y \in Y\}$ is called the filter trace of Y on X.

Y is called a prime extension of X if $\mathcal{O}_y$ is a prime open filter for each $y \in Y$. Note: $\mathcal{O}_x$ is always a prime open filter for $x \in X$.

Let t(strict) be the topology on Y generated by the base $\{0^*: 0 \in t(X)\}$, where $0^* = \{y \in Y: 0 \in \mathcal{O}_y\}$. Let t(simple) be the topology on Y generated by the base $\{0 \cup \{y\}: 0 \in \mathcal{O}_y, y \in Y\}$. Then t(strict) and t(simple) are such that Y with either of these topologies is an extension of (X,t(x)), called a strict extension, or simple extension of X, respectively. Note that

$$t(\text{strict}) \le t \le t(\text{simple}).$$

Moreover, a topology s on Y with the same filter trace as t forms an extension of $(X,t(X))$ if and only if it satisfies the above inequality. (See Banaschewski [1].)

Herrlich's completion of a nearness space was presented in [9]. A brief description of it appears in [2] which we provide here for the convenience of the reader. Let (X,ξ) be a nearness space and let Y be the set of all ξ-clusters $\mathcal{A}$ with empty adherence. Set $X^* = X \cup Y$. For each $A \subset X$, define $cl(A) = \{y \in Y: A \in y\} \cup cl_\xi A$. A nearness structure ξ^* is defined on X^* as follows: $\mathcal{B} \in \xi^*$ provided $\mathcal{A} = \{A \subset X:$ there exists $B \in \mathcal{B}$ with $B \subset cl(A)\} \in \xi$. (X^*,ξ^*) is a complete nearness space with $cl_{\xi^*}X = X^*$. Also, for $A \subset X$, $cl_{\xi^*}A = cl(A)$.

The following important theorem is due to Herrlich and Bently [2].

Theorem A. *For any* T_1 *nearness space* (X,ξ) *the following conditions are equivalent.*

(1) ξ *is a nearness structure induced on* X *by a strict extension.*

(2) *The completion* (X^*,ξ^*) *of* (X,ξ) *is topological.*

(3) (X,ξ) *is concrete.*

It is shown in [4] that a concrete nearness structure is ultrafilter generated if and only if it is induced by a prime strict extension. The following theorem also appears in [4].

Theorem B. *For any* T_1 *nearness space the following conditions are equivalent.*

(1) ξ *is induced on* X *by a prime strict extension.*

(2) *The completion* (X^*, ξ^*) *of* (X, ξ) *is topological and* X^* *is a prime extension of* X.

(3) ξ *is concrete and ultrafilter generated.*

Since the Pervin nearness structure is ultrafilter generated and concrete it follows that ξ_p is induced by a prime strict extension, provided the underlying topology is T_1. We now show that for T_1 spaces the Pervin nearness structure is induced by the Wallman compactification.

Let (X, ξ) be a nearness space and set $\xi_c = \{\mathcal{A} \subset \mathcal{P}(X):$ each finite $\mathcal{B} \subset \mathcal{A}$ belongs to $\xi\}$. The following theorem is due to Herrlich [10].

Theorem C. *If* (X, ξ) *is* T_1 *and topological then* (X^*, ξ_c^*) *is the Wallman compactification.*

If ξ is topological then $\xi_c = \{\mathcal{A} \subset \mathcal{P}(X): \mathcal{A}$ has f.i.p.$\}$ $= \xi_p$. Thus the following theorm is an immediate consequence of Herrlich's result.

Theorem D. *Let* (X, t) *be a* T_1 *topological space. Let* ξ_p *be the Pervin nearness structure. Then* (X^*, ξ_p^*) *is the Wallman compactification of* X.

Since the Wallman compactification of a normal space is the Stone-Čech compactification we have the following corollary.

Corollary 4.1. *Let* (X,t) *be a normal topological space and* ξ_p *the Pervin nearness structure. Then* (X^*, ξ_p^*) *is* βX, *the Stone-Čech compactification of* X.

Corollary 4.2. *Let* (X,t) *be a* T_1 *topological space. Then the Wallman compactification is a prime extension of* X. *If* X *is normal then* βX *is a prime extension of* X.

Corollary 4.3. *Let* (X,t) *be a* T_1 *topological space. Then the Pervin nearness structure is the nearness structure on* X *induced by the Wallman compactification of* X. *Similarly, if* X *is normal then the Pervin nearness structure is the nearness structure induced by the Stone-Čech compactification of* X.

By corollary 4.3 and the results obtained for separated and regular nearness spaces obtained in [2], we have the following theorem.

Theorem 4.4. *Let* (X,t) *be a* T_1 *topological space. The following statements are equivalent.*

(1) X *is normal.*

(2) ξ_p *is separated.*

(3) ξ_p *is regular.*

Theorem 4.5. *Let* (X,ξ) *be a* T_1 *nearness space and* (X^*, ξ^*) *its completion. Then the trace filters on* X *are given by:*

(1) $\mathcal{O}_x = \{0 \in t(\xi): x \in 0\}$ *for* $x \in X$, *and*

(2) $\mathcal{O}_{\mathcal{A}} = \{0 \in t(\xi): X-0 \notin \mathcal{A}\}$ *for* $\mathcal{A} \in X^*-X$.

Proof. (1) follows immediately since X is a dense subspace of X*. (2). Let $\mathcal{A}$ be a ξ-cluster and set $\mathcal{S} = \{0 \in t(\xi): X-0 \notin \mathcal{A}\}$. Let $0 \in \mathcal{S}$. Then $X-0 \notin \mathcal{A}$. Now cl(X-0) is closed in X* and $\mathcal{A} \notin$ cl(X-0). Thus $Q^* = X^* -$ cl(X-0) is open in X* and $0 = Q^* \cap X$ and $\mathcal{A} \in Q^*$. Thus $0 \in \mathcal{O}_{\mathcal{A}}$ and thus $\mathcal{S} \subset \mathcal{O}_{\mathcal{A}}$.

Let $0 \in \mathcal{O}_{\mathcal{A}}$. Then there exists Q*, open in X*, with $\mathcal{A} \in Q^*$ such that $0 = X \cap Q^*$. Now cl(X-0) is closed in X* but $\mathcal{A} \in Q^*$ and $Q^* \cap (X-0) = \phi$. Hence $\mathcal{A} \in$ cl(X-0) and thus $X-0 \in \mathcal{A}$. Therefore, $0 \in \mathcal{S}$ and $\mathcal{O}_{\mathcal{A}} \subset \mathcal{S}$.

Theorem 4.6. *Let* (X,t) *be a* T_1 *topological space and* ξ_p *the Pervin nearness structure. Then the trace filters generated by the completion* (X^*, ξ_p^*) *are precisely the minimal prime open filters on* X.

Proof. By theorem 4.5, the trace filters for the completion are of the form:

(1) $\mathcal{O}_x = \{0 \in t: x \in 0\}$, for $x \in X$; or

(2) $\mathcal{O}_{\mathcal{A}} = \{0 \in t: X-0 \notin \mathcal{A}\}$ for $\mathcal{A}$ a ξ_p-cluster.

By lemma 2.3, each $\mathcal{O}_x$ is a minimal prime open filter. If $\mathcal{A}$ is a ξ_p-cluster then by theorem 3.7 there exists a closed ultrafilter $\mathcal{F}$ such that $\mathcal{A} = \mathcal{G}(\mathcal{F})$. Thus $\mathcal{O}_{\mathcal{A}} = \{0 \in t: X-0 \notin \mathcal{G}(\mathcal{F})\} = \{0 \in t: X-0 \notin \mathcal{F}\}$. Thus, by lemma 2.3, $\mathcal{O}_{\mathcal{A}}$ is a minimal prime open filter.

Thus each $\mathcal{O}_x$ and $\mathcal{O}_{\mathcal{A}}$ is a minimal prime open filter; all that remains to be shown is that if $\mathcal{O}$ is a minimal prime open filter then either $\mathcal{O} = \mathcal{O}_x$ for some $x \in X$ or $\mathcal{O} = \mathcal{O}_{\mathcal{A}}$ for some ξ_p-cluster $\mathcal{A}$.

Let $\mathcal{O}$ be a minimal prime open filter. Then, by lemma 2.3, there exists a closed ultrafilter $\mathcal{F}$ such that $\mathcal{O} = \mathcal{O}(\mathcal{F})$.

If $\{x\} \in \mathcal{F}$ then $\mathcal{O}_x \subset \mathcal{O}$ and, since $\mathcal{O}$ is a minimal prime open filter, it follows that $\mathcal{O} = \mathcal{O}_x$.

If $\cap\mathcal{F} = \phi$ then $\mathcal{G}(\mathcal{F}) \in X^*-X$. Then $\mathcal{O}_{\mathcal{G}(\mathcal{F})} = \{0 \in t: X-0 \notin \mathcal{G}(\mathcal{F})\} = \{0 \in t: X-0 \notin \mathcal{F}\} = \mathcal{O}(\mathcal{F}) = \mathcal{O}$.

Corollary 4.7. *The family of minimal prime open filters is the filter trace of the Wallman compactification for a* T_1 *space; and correspondingly, the filter trace of the Stone-Čech compactification for a normal space.*

Let (X,t) be a topological space and Y' a collection of open filters on X containing the open neighborhood filters. For $0 \in t$, let $0^* = \{\mathcal{O} \in Y': 0 \in \mathcal{O}\}$. Let t' be the topology on Y' generated by the base $\{0^*: 0 \in t\}$. Then, by Banaschewski [1], (Y',t') is a strict extension of X where the embedding map $e: X \to Y'$ is given by $e(x) = \mathcal{O}_x$, the open neighborhood filter of x.

Many well known extensions can be constructed in this manner. The following examples, among others, are given in [1].

(1) If (X,t) is completely regular and Y' is the family of all maximal completely regular open filters then (Y',t') is the Stone-Čech compactification.

(2) If (X,t) is Hausdorff and Y' is the family of all open ultrafilters then (Y',t') is the Fomin H-closed extension of X.

(3) If (X,t) is a locally compact, non-compact Hausdorff space and Y' is the collection of all open neighborhood

filters together with the filter of open sets with compact complements then (Y',t') is the 1-point Alexandroff compactification of X.

By theorems D and 4.6, we have that the Wallman compactification and the Stone-Čech compactification for normal spaces can be constructed in this manner, using the family of all minimal prime open filters.

Theorem 4.8. *Let* (X,t) *be a* T_1 *topological space and let* Y' *be the family of all minimal prime open filters on* X. *Then:*

(1) (Y',t') *is the Wallman compactification of* X.

(2) *If* (X,t) *is normal then* (Y',t') *is the Stone-Čech compactification of* X.

In [6], nearness spaces whose completions were second category or Baire topological spaces were characterized. In order to relate these results to the Pervin nearness structure we state a definition and two theorems from [6].

Definition 4.1. Let (X,ξ) be a nearness space. Let $\mathcal{S} \subset \mathcal{P}(X)$. Then $\mathcal{S}$ is called a sparce near collection if $\mathcal{S} \in \xi$ and for each $\mathcal{B} \in \xi$, such that $\mathcal{B}$ is not contained in each ξ-cluster, then there exists a ξ-cluster $\mathcal{A}$ such that $\mathcal{B} \not\subset \mathcal{A}$ and $\mathcal{S} \not\subset \mathcal{A}$.

Let $\{\mathcal{S}_\alpha : \alpha \in I\} \subset \xi$. Then $\{\mathcal{S}_\alpha : \alpha \in I\}$ is called a uniformly sparce family if for each $\mathcal{B} \in \xi$, such that $\mathcal{B}$ is not contained in each ξ-cluster, then there exists a ξ-cluster $\mathcal{A}$ such that $\mathcal{B} \not\subset \mathcal{A}$ and $\mathcal{S}_\alpha \not\subset \mathcal{A}$ for each $\alpha \in I$.

Easily each member of a uniformly sparce family is itself a sparce near collection.

Theorem 4.9 (Carlson [6]). *Let* (X,ξ) *be a* T_1 *nearness space. The following statements are equivalent.*

(1) ξ *is a nearness structure induced on* X *by a second category* T_1 *strict extension.*

(2) *The completion* (X^*,ξ^*) *of* (X,ξ) *is topological and second category.*

(3) (X,ξ) *is concrete and for each countable collection* $\{\mathcal{S}_i : i \in N\}$ *of sparce near collections there exists a* ξ*-cluster* $\mathcal{A}$ *such that* $\mathcal{S}_i \cup \mathcal{A} \notin \xi$ *for each* $i \in N$.

Theorem 4.10 (Carlson [6]). *Let* (X,ξ) *be a* T_1 *nearness space. The following statements are equivalent.*

(1) ξ *is a nearness structure induced on* X *by a strict* T_1 *Baire extension.*

(2) *The completion* (X^*,ξ^*) *of* (X,ξ) *is topological and a Baire space.*

(3) (X,ξ) *is concrete and each countable family of sparce near collections is uniformly sparce.*

Theorem 4.11. *Let* (X,t) *be a* T_1 *topological space and* ξ_p *the Pervin nearness structure on* X. *Let* $\mathcal{A} \in \xi_p$.

(1) $\mathcal{A}$ *is not contained in each* ξ_p*-cluster if and only if* $\mathcal{F} = \mathcal{F}(\overline{\mathcal{A}})$ *is not the trivial closed filter.*

(2) $\mathcal{A}$ *is a sparce near collection if and only if* $\mathcal{F} = \mathcal{F}(\overline{\mathcal{A}})$ *is a sparce closed filter.*

Proof. (1) follows from the fact that $\mathcal{A}$ is contained in each ξ_p-cluster if and only if each $A \in \mathcal{A}$ is dense in X.

(2) Suppose $\mathcal{A}$ is a sparce near collection. Then $\mathcal{F} = \mathcal{F}(\overline{\mathcal{A}})$ is a closed filter. Let $\mathcal{H}$ be a nontrivial closed filter. Then $\mathcal{H} \in \xi_p$ and $\mathcal{H}$ is not contained in each ξ_p-cluster. Since $\mathcal{A}$ is a sparce near collection there exists a ξ_p-cluster $\mathcal{B}$ such that $\mathcal{A} \not\subset \mathcal{B}$ and $\mathcal{H} \not\subset \mathcal{B}$. Now there exists a closed ultrafilter $\mathcal{G}$ such that $\mathcal{B} = \{B \subset X: \overline{B} \in \mathcal{G}\}$. Easily $\mathcal{F} \not\subset \mathcal{G}$ and $\mathcal{H} \not\subset \mathcal{G}$. Therefore, $\mathcal{F}$ is a sparce closed filter.

Now suppose $\mathcal{F} = \mathcal{F}(\mathcal{A})$ is a sparce closed filter. Let $\mathcal{H} \in \xi_p$ and $\mathcal{H}$ not contained in each ξ_p-cluster. Then $\mathcal{M} = \mathcal{F}(\overline{\mathcal{H}})$ is a nontrivial closed filter. Since $\mathcal{F}$ is a sparce closed filter there exists a closed ultrafilter $\mathcal{G}$ such that $\mathcal{F} \not\subset \mathcal{G}$ and $\mathcal{M} \not\subset \mathcal{G}$. Let $\mathcal{B} = \{B \subset X: \overline{B} \in \mathcal{G}\}$. Then $\mathcal{B}$ is a ξ_p-cluster and $\mathcal{M} \not\subset \mathcal{B}$ and $\mathcal{A} \not\subset \mathcal{B}$. Hence $\mathcal{A}$ is a sparce near collection.

Theorem 4.12. *Let* (X,t) *be a* T_1 *topological space. The following statements are equivalent.*

(1) *The Wallman compactification of* X *is second category.*

(2) *For each countable collection* $\{\mathcal{F}_i: i \in N\}$ *of sparce closed filters there exists a closed ultrafilter* $\mathcal{G}$ *such that* $\mathcal{F}_i \not\subset \mathcal{G}$ *for each* $i \in N$.

Proof. Let ξ_p denote the Pervin nearness structure on X. Then (X^*,ξ_p^*) is the Wallman compactification of X and the result holds by theorems 4.9 and 4.11.

Theorem 4.13. *Let* (X,t) *be a* T_1 *topological space. The following statements are equivalent.*

(1) *The Wallman compactification of X is a Baire space.*

(2) *Each countable family of sparce closed filters is uniformly sparce.*

Proof. Let ξ_p denote the Pervin nearness structure on X. Then (X^*, ξ_p^*) is the Wallman compactification of X and the result holds by theorems 4.10 and 4.11.

Corollary 4.14. *Let* (X,t) *be a normal topological space. Then each countable family of sparce closed filters is uniformly sparce.*

Proof. This follows immediately from the fact that each compact Hausdorff space is a Baire space.

5. Balanced Near Collections

Definition 5.1. Let (X,ξ) be a nearness space. Let $\mathcal{A} \subset \mathcal{P}(X)$ and let $c(\mathcal{A})$ denote the set of all ξ-clusters that contain $\mathcal{A}$. Let $\rho(\mathcal{A})$ denote the set of all the fixed ξ-clusters that contain $\mathcal{A}$. Set $b(\mathcal{A}) = \cap c(\mathcal{A})$. $\mathcal{A}$ is said to be a balanced near collection provided $\mathcal{A} \in \xi$ and $\mathcal{A} = \cap c(\mathcal{A})$ and $\mathcal{A}$ is called a rigid near collection provided $\mathcal{A} \in \xi$ and $\mathcal{A} = \cap\rho(\mathcal{A})$.

Theorem 5.1. *Let* (X,ξ) *be a concrete nearness space and* $\mathcal{A} \in \xi$.

(1) $\mathcal{A} \subset \cap c(\mathcal{A})$.

(2) $b(\mathcal{A})$ *is the smallest balanced near collection containing* $\mathcal{A}$.

(3) *Each* ξ*-cluster is a balanced near collection.*

(4) *If* η *is a nonempty collection of* ξ*-clusters then* $\cap\eta$ *is a balanced near collection.*

(5) *If* $\mathcal{A}$ *is a rigid near collection then* $\mathcal{A}$ *is a balanced near collection.*

Let Y be a strict T_1 extension of X and ξ the nearness structure on X induced by Y. For $y \in Y$, let $\mathcal{A}_y = \{A \subset X: y \in cl_Y A\}$. Then $\{y\} = \cap cl_Y \mathcal{A}_y$ and each ξ-cluster is of the form $\mathcal{A}_y$ for some $y \in Y$.

For $\phi \neq F \subset Y$, let $\mathcal{S}(F) = \{A \subset X: F \subset cl_Y A\}$. We shall see that the balanced near collections are precisely of this form. Since the extension is strict it follows that $cl_Y F = \cap cl_Y \mathcal{S}(F)$.

Theorem 5.2. *Let* Y *be a* T_1 *strict extension of* X. *Set* $\xi = \{\mathcal{A} \subset \mathcal{P}(X): \cap cl_Y \mathcal{A} \neq \phi\}$.

(1) *If* $\phi \neq F \subset Y$ *then* $c(\mathcal{S}(F)) = \{\mathcal{A}_y: y \in cl_Y F\}$ *and* $\mathcal{S}(F)$ *is a balanced near collection.*

(2) *If* $\mathcal{A}$ *is a balanced near collection then there exists a closed set* F *in* Y *such that* $\mathcal{A} = \mathcal{S}(F)$.

(3) *If* $\phi \neq F \subset Y$ *then* $\mathcal{S}(F) = \mathcal{S}(cl_Y F)$.

(4) *If* F *and* E *are nonempty subsets of* Y *then* $\mathcal{S}(F \cup E) = \mathcal{S}(F) \cap \mathcal{S}(E)$.

(5) *There exists a one-to-one correspondence between the nonempty closed subsets of* Y *and the balanced near collections.*

Theorem 5.3. *Let* Y *be a strict* T_1 *extension of* X *and* ξ *the nearness structure on* X *induced by* Y. *Let* $\mathcal{A} \subset \mathcal{P}(X)$.

(1) *If* $T = \{x: \mathcal{A}_x \in \rho(\mathcal{A})\}$ *then* $T = \cap cl_X \mathcal{A}$.

(2) $\mathcal{A}$ *is a rigid near collection if and only if there exists a nonempty closed set* T *in* X *such that* $\mathcal{A} = \{A \subset X: T \subset cl_X A\}$.

(3) $\mathcal{A}$ *is a rigid near collection if and only if*

(A) $\mathcal{A}$ *is a balanced near collection, and*

(B) $cl_Y(\cap cl_X \mathcal{A}) = \cap cl_Y \mathcal{A}$.

(4) *Let* F *be a nonempty closed subset of* Y. *The following statements are equivalent.*

(A) $F = cl_Y(F \cap X)$

(B) *There exists a unique closed subset of* X, *say* G, *such that* $F = cl_Y G$.

(C) $\mathcal{S}(F)$ *is a rigid near collection.*

We now characterize the balanced near collections in a Pervin nearness space.

Theorem 5.4. *Let* (X,t) *be a* T_1 *topological space and* ξ_p *the Pervin nearness structure on* X. *Let* $\mathcal{A} \subset \mathcal{P}(X)$.

(1) $\mathcal{A}$ *is a balanced near collection in* ξ_p *if and only if there exists a balanced closed filter* $\mathcal{F}$ *such that* $\mathcal{A} = \mathcal{G}(\mathcal{F})$.

(2) *There exists a one-to-one correspondence between the nonempty closed sets in* (X^*, ξ_p^*) *and the balanced closed filters in* X. *Moreover; each nonempty closed set* $F^* \subset X^*$ *is of the form* $F^* = \cap cl_{X^*} \mathcal{F} = \{\mathcal{G}(\mathcal{M}): \mathcal{M}$ *is a closed ultrafilter containing* $\mathcal{F}\}$, *where* $\mathcal{F}$ *is a balanced closed filter on* X.

(3) $\mathcal{A}$ *is a rigid near collection in* ξ_p *if and only if there exists a rigid closed filter* $\mathcal{F}$ *such that* $\mathcal{A} = \mathcal{G}(\mathcal{F})$.

Corollary 5.5. *Let* (X,t) *be a* T_1 *(Normal) topological space and* ωX *its Wallman compactification* (βX *its Stone-Čech compactification). Then there exists a one-to-one correspondence between the nonempty closed sets in* ωX (βX) *and the balanced closed filters in* X. *Specifically, if* F *is a nonempty closed set in* ωX (βX) *then there exists a unique balanced closed filter* $\mathcal{F}$ *such that* $F = \cap cl_{\omega X}\mathcal{F}$ ($F = \cap cl_{\beta X}\mathcal{F}$).

6. βN

Let N denote the natural numbers with the discrete topology and let ξ_p denote the Pervin nearness structure on N. Since N is normal, $(N^*, \xi_p^*) \cong \beta N$. Now the points of βN are the ξ_p-clusters; that is, they are of the form $\mathcal{G}(\mathcal{M})$, where $\mathcal{M}$ is an ultrafilter. Since the topology is discrete it follows that $\mathcal{G}(\mathcal{M}) = \mathcal{M}$. (Throughout this section, $\mathcal{M}$ will denote an ultrafilter on N.)

Thus, the points of βN are simply the ultrafilters on N. Since N* is a strict extension of N it follows that the sets of the form $cl_{N^*}E$, for $E \subset N$, form a base for the closed sets of βN.

Now, $cl_{N^*}E = \{\mathcal{M}: E \in \mathcal{M}\}$, which we will call V(E). This agrees with a standard construction of βN; see for example, M. E. Rudin [11]. It is also known that the open-closed subsets of βN are of the form V(E) for some $E \subset N$. Moreover; $\{V(E): E \subset N\}$ is a base for the open sets and a base for the closed sets in βN.

Lemma 6.1. *Every filter on* N *is a balanced closed filter.*

By corollary 5.6, there exists a one-to-one correspondence between the nonempty closed subsets of (N^*, ξ_p^*) and the balanced closed filters on (N, ξ_p). In light of lemma 6.1, we have that there exists a one-to-one correspondence between the nonempty closed subsets of βN and the filters on N. The following theorem provides a characterization of the nonempty closed subsets of βN.

Theorem 6.2. *Let* $\phi \neq F \subset \beta N$. F *is closed if and only if there exists a filter* $\mathcal{F}$ *on* N *such that* $F = \{\mathcal{M} \in \beta N: \mathcal{F} \subset \mathcal{M}\}$.

Proof. If F is closed and nonempty in $\beta N = N^*$ then $F = \cap cl_{N^*}\mathcal{S}(F)$. By theorem 5.2, $\mathcal{S}(F)$ is a balanced near collection and by theorem 5.4 and lemma 6.1 there exists a filter $\mathcal{F}$ on N such that $\mathcal{S}(F) = \mathcal{G}(\mathcal{F}) = \mathcal{F}$. Now, for $F \subset N$, $cl_{N^*}F = \{\mathcal{M}: F \in \mathcal{M}\}$. Hence $F = \cap cl_{N^*}\mathcal{S}(F) = \cap cl_{N^*}\mathcal{F} = \{\mathcal{M}: \mathcal{F} \subset \mathcal{M}\}$.

On the other hand, if $\mathcal{F}$ is a filter on N, then $F = \cap cl_{N^*}\mathcal{F}$ is a closed set in $N^* = \beta N$.

Claim. $F = \{\mathcal{M}: \mathcal{F} \subset \mathcal{M}\}$. Suppose $\mathcal{M} \in F = \cap cl_{N^*}\mathcal{F}$. Then $A \in \mathcal{M}$ for each $A \in \mathcal{F}$. That is, $\mathcal{F} \subset \mathcal{M}$. Conversely, if $\mathcal{F} \subset \mathcal{M}$ then $\mathcal{M} \in cl_{N^*}A$ for each $A \in \mathcal{F}$. Thus $\mathcal{M} \in \cap cl_{N^*}\mathcal{F} = F$.

Corollary 6.3. *If* O *is an open subset of* βN *and* $O \neq \beta N$, *then there exists a filter* $\mathcal{F}$ *on* N *such that* $O = \{\mathcal{M}: \mathcal{F} \not\subset \mathcal{M}\}$.

Corollary 6.4. *If* A *is a nontrivial open-closed subset of* βN *then there exists filters* $\mathcal{F}$ *and* $\mathcal{H}$ *on* βN *such that*

$$A = \{\mathcal{M}: \mathcal{F} \subset \mathcal{M}\} = \{\mathcal{M}: \mathcal{H} \not\subset \mathcal{M}\}.$$

Thus, if A is a nontrivial open-closed subset of βN, there must exist a pair of filters $\mathcal{J}$ and $\mathcal{H}$ on N such that each ultrafilter on N contains one and only one of these filters. It is easy to show that the only possibility is for $\mathcal{J} = \{E \subset N: S \subset E\}$ and $\mathcal{H} = \{E \subset N: N-S \subset E\}$ for some $\phi \neq S \neq N$. Thus, the nontrivial open-closed subsets of βN are simply the $V(S) = \{\mathcal{M}: S \in \mathcal{M}\}$ as stated in [11].

Theorem 6.5. *There exists a natural one-to-one correspondence between the following classes of filters on* N *and the respective special nonempty closed subsets of* βN.

	N	βN
(1)	*ultrafilters*	*points*
(2)	*filters*	*closed sets*
(3)	$\mathcal{J}$(A) *filters*	*open-closed sets*
(4)	*filters with countable base*	*closed* G_δ *sets*
(5)	*filters with countable base*	*zero sets*

Proof. The correspondence is as discussed; namely, $F = \{\mathcal{M}: \mathcal{J} \subset \mathcal{M}\}$ where F is a closed set in βN and $\mathcal{J}$ is a filter on N. The correspondences for (1), (2) and (3) have already been noted.

Proof of (4) and (5). In a normal space, a subset F is a closed G_δ set if and only if it is a zero set.

Let F be a nonempty zero set in βN. From (6E) in [8], we have that every zero set in βX is a countable intersection of sets of the form $cl_{\beta X}Z$ where Z is a zero set in X. Since every subset of N is a zero set in N it follows that $F = \cap\{cl_{\beta N}F_i: F_i \subset N \text{ and } i \in N\}$. Since F is nonempty, it follows that there exists an ultrafilter containing

$\{F_i: i \in N\}$. Hence the F_i have the f.i.p. and we let $\mathcal{F}$ denote the filter generated by these sets. Then $\mathcal{F}$ has a countable base and moreover $F = \{\mathcal{M}: \mathcal{F} \subset \mathcal{M}\}$.

Now let $\mathcal{F}$ be a filter with a countable base, say $\{G_i: i \in N\}$, and $F = \{\mathcal{M}: \mathcal{F} \subset \mathcal{M}\}$. Then $F = \cap\{cl_{\beta N}G_i: i \in N\}$ and hence F is a zero set in βN.

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