TOPOLOGY PROCEEDINGS

Volume 9, 1984

Pages 31–36

http://topology.auburn.edu/tp/

MORE ON CAUCHY CONDITIONS

by

S. W. Davis

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

MORE ON CAUCHY CONDITIONS

S. W. Davis

0. Introduction

This note is a continuation of [D]. The question which prompts this work was asked by Arhangel'skiĭ and relayed by M. E. Rudin, namely, "Is every Lindelöf symmetrizable space separable?" This question appears in print in [NC].

The approach which has been used on this problem by Kofner [K], Nedev [N] and this author has been to increase the strength of the symmetric involved by adding a Cauchy-like requirement for convergent sequences. These conditions are described in detail in [D], and we will not repeat them here with the exception of the condition which we will be using.

Definition 0.1 [A]. A space X is called *weakly first countable* iff there exists $B: \omega \times X \to \mathcal{P}(X)$ such that (i) for all $x \in X$ we have $B(n+1,x) \subseteq B(n,x)$ for all $n \in \omega$, and $\bigcap_{n\in\omega} B(n,x) = \{x\}$ and (2) a set $U \subseteq X$ is open iff for each $x \in U$ there exists $n_x \in \omega$ such that $B(n_x,x) \subseteq U$.

The function B in 0.1 is called a wfc-system which is compatible with X.

Definition 0.2. If B is a wfc-system for a space X, then we say B is wC iff whenever $A \subseteq X$ and there exists $n \in \omega$ such that $B(n,x) \cap A = \{x\}$ for all $x \in A$, then A is relatively discrete.

The present paper concerns wC wfc-systems on symmetrizable spaces. We show that if a Lindelöf symmetrizable space has a wC wfc-system, then it is separable. We also present an example of a symmetrizable space with no wC wfc-system which answers a question raised in [D].

We use the notation $\omega = \{0,1,2,\cdots\}$ and $\mathbb{N} = \{1,2,\cdots\}$.

1. The Main Result

In this section, we give the (surprisingly easy) proof of our main theorem. First, we give a lemma which is quite useful in working on these properties.

Lemma 1.1. *Suppose* X *is a* T_2*-space,* d *is a symmetric compatible with* X, *and* B *is a* wfc*-system compatible with* X. *For each* $x \in X$ *and* $n \in \mathbb{N}$, *there exists* $k \in \omega$ *such that* $B(k,x) \subseteq \{y: d(x,y) < \frac{1}{n}\}$.

Proof. If the theorem is false, then choose $x \in X$ and $n \in \mathbb{N}$ witnessing that fact. For each $k \in \omega$ choose $x_k \in B(k,x)$ with $d(x,x_k) \geqq \frac{1}{n}$. Since B is compatible with X, the sequence $\langle x_k: k \in \omega \rangle$ converges to x. Consider the set $\{x_k: k \in \omega\}$. Suppose $y \notin \{x_k: k \in \omega\}$. If $y = x$, then $d(y,\{x_k: k \in \omega\}) \geqq \frac{1}{n} > 0$. If $y \neq x$, then $\{x\} \cup \{x_k: k \in \omega\}$ is a closed set excluding y, thus $d(y,\{x_k: k \in \omega\}) \geqq d(y,\{x\} \cup \{x_k: k \in \omega\}) > 0$. In either case then $d(y,\{x_k: k \in \omega\}) > 0$ for all $y \in X\setminus\{x_k: k \in \omega\}$. Since d is compatible with X, this says that $\{x_k: k \in \omega\}$ is closed which is clearly impossible since $x \in \overline{\{x_k: k \in \omega\}}\setminus\{x_k: k \in \omega\}$. Hence we have the result.

Remark. Although we shall not need it here, a somewhat stronger result can be obtained; namely, if X is T_2 and B, B' are wfc-systems compatible with X, then for each $x \in X$ and $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $B(k,x) \subseteq B'(n,x)$.

Lemma 1.2 [Nedev]. *Every regular Lindelöf symmetrizable space is hereditarily Lindelöf.*

Besides being useful for the next result, 1.2 shows that a counterexample (if there is one) to the main question must be an L-space.

Theorem 1.3. *If* X *is a regular Lindelöf symmetrizable space with a compatible* wC wfc-*system, then* X *is separable.*

Proof. Let d be a symmetric compatible with X, and B a wC wfc-system compatible with X. For each $n \in \mathbb{N}$, let $D_n \subseteq X$ be maximal with respect to $d(x,y) \geqq \frac{1}{n}$ for $x,y \in D_n$, $x \neq y$. Now $D = \cup_{n=1}^{\infty} D_n$ is dense in X, since if $x \in U$ with U open and $U \cap D = \emptyset$, then choose $k \in \mathbb{N}$ with $d(x,X\backslash U) \geqq \frac{1}{k}$, and D_k is not maximal. Fix $n \in \mathbb{N}$. For each $x \in D_n$, by Lemma 1.1, we let $k(x,n)$ be an element of the set $\{k: B(k,x) \subseteq \{y: d(x,y) < \frac{1}{n}\}\}$. For each $k \in \omega$, let $E_k = \{x \in D_n: k(x,n) = k\}$. For each $x \in E_k$, $B(k,x) \cap E_k = \{x\}$. Since B is wC, E_k is then relatively discrete. By Lemma 1.2, X is hereditarily Lindelöf, so E_k is countable. Now since $D_n = \cup_{k \in \omega} E_k$, we have that D_n is countable. This is true for each $n \in \mathbb{N}$, thus D is the desired countable dense subset.

2. The Main Example

In this section, we describe an old example of Kofner [K], and show that it has no compatible wC wfc-system even though it is a separable symmetrizable space. This answers negatively Question 4.1 of [D].

Example 2.1 [Kofner]. Let $\mathbb{R}$ denote the real numbers, I denote the irrational numbers, and $\mathbb{Z}$ denote the integers. Let $D_0 = \emptyset$, and for $i \in \mathbb{N}$, let $D_i = \{\frac{k}{2^{i-1}}: k \in \mathbb{Z}\}$. Let $D = \bigcup_{i\in\omega} D_i$. Note that D is the set of dyadic rationals. Let $M = I \cup D$. For $x \in M$ and $j \in \mathbb{N}$, we let $T_j(x) = M \cap (x - \frac{1}{2^{j-1}}, x + \frac{1}{2^{j-1}})$. For $x \in I$, let $S_j(x) = T_j(x) \cap D_j$ for each $j \in \mathbb{N}$. Let $0_n(x) = \bigcup_{j\geq n} S_j(x) \cup \{x\}$ for each $n \in \mathbb{N}$.

For $x \in D$, choose i with $x \in D_i \setminus D_{i-1}$. For $n \in \mathbb{N}$, let $m = \max\{i,n\}$, let $U_n(x) = T_n(x) \cap D$, and let $0_n(x) = T_m(x) \cup U_n(x)$.

Define $B^*: \mathbb{N} \times M \to \mathcal{P}(M)$ by $B^*(n,x) = 0_n(x)$. Then using (2) of Definition 0.1 to define a topology on M, B* is a wfc-system.

Let X be the space whose underlying set is M with the topology generated by B*.

Lemma 2.2 [Kofner]. B* *is a symmetric* wfc-*system (i.e.* $x \in B^*(n,y) \Rightarrow y \in B^*(n,x)$*) for the topology on* X *which is finer than Euclidean. Hence* X *is* T_2 *and symmetrizable.*

Lemma 2.3 [Kofner]. X *is zero dimensional. Hence* X *is a Tychonoff space.*

Theorem 2.4. *There is no* wC wfc-*system which is compatible with* X.

Proof. Suppose B is any wfc-system which is compatible with X. For $n \in \omega$, let $I_n = \{x \in I: B(n,x) \cap I = \{x\}\}$. Since for each $x \in I$, there exists $k \in \omega$ such that $B(k,x) \subseteq 0_1(x)$ by Lemma 1.1, we have that $\cup_{n\in\omega} I_n = I$. Since I is second category in $\mathbb{R}$, choose $n \in \omega$ and an open interval (a,b) such that $a \notin M$, $b \notin M$, and $I_n \cap (a,b)$ is dense in the Euclidean topology on (a,b). Now in the topology of X, each point of $I_n \cap (a,b)$ is a cluster point of I_n so that I_n can not be relatively discrete. Hence B is not wC.

3. Sequential Separability and Questions

The positive results on the question we've been considering all have a similarity in the way the countable dense subset is constructed. Indeed, the countable dense set D is such that each point in the space is the limit of a sequence drawn from D. In non-Fréchet spaces, this "sequential separability" seems too much to expect.

Question 3.1. Is every separable symmetrizable space "sequentially separable"?*

The main question could still be answered by an affirmative answer to the following:

Question 3.2. Does every Lindelöf symmetrizable space admit a wC wfc-system?

---

*See Note added in proof.

Related to wC and weak Cauchy conditions (see [D] for definitions), the following questions remain.

Question 3.3. If X has a symmetric and a wC wfc-system, must X have a wC symmetric?

Question 3.4. If X has a wC symmetric, must X have a weakly Cauchy symmetric?

Kofner [K] has shown that the answer is "yes" to 3.3 if "wC" is replaced by "weakly Cauchy." Example 2.5 of [D] shows that 3.4 does not trivially have a "yes" answer.

Note (added in proof). L. Foged has given an example answering 3.1 in the negative.

References

[A] A. V. Arhangel'skiǐ, *Mappings and spaces*, Russian Math. Surveys 21 (1966), 115-162.

[D] S. W. Davis, *Cauchy conditions on symmetrics*, Proc. Amer. Math. Soc. 86 (1982), 349-352.

[K] J. A. Kofner, *Symmetrizable spaces and factor mappings*, translated from Matematicheskie Zametki 14 (1973), 713-722.

[N] S. I. Nedev, *Symmetrizable spaces and final compactness*, Soviet Math. Dokl. 8 (1967), 890-892.

[NC] ________ and M. M. Choban, *On the theory of* o-*metrizable spaces, II*, Vestnik Moskovskogo Universiteta. Matematika, 27 (2) (1972), 10-17.

Miami University

Oxford, Ohio 45056