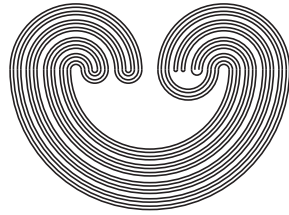

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ON UNIFORM HYPERSPACES

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ON UNIFORM HYPERSPACES

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1. Introduction

This paper is a continuation of the work done during the 1970's by Z. Frolík, A. Hager, M. Hušek, J. Pelant, M. Rice and others on uniform spaces. While the topological study of hyperspaces is a thriving part of mathematics, in the field of uniform topology the uniform hyperspaces have been left essentially unstudied except for John Isbell's characterization of uniform spaces whose uniform hyperspace of all nonempty closed subsets is a complete uniform space, see [7]. Our aim is to fill a part of this gap by considering questions related to recent research on uniform spaces. In addition to other results, we characterize the class of uniform spaces whose hyperspaces are metric-fine and the result follows from the fact that the metric-completion of Morita and Rice commutes with the operation of forming the uniform hyperspace of all nonempty compact subsets.

2. Some Preliminary Definitions

The reader may consult [8] for information on uniform spaces. A set X with a uniformity μ is called a uniform space and denoted by μX . In this paper all uniform spaces

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will be separated. If ρ is a pseudometric on a set X , then $B_{\rho, \varepsilon}(x)$ denotes the set of all points y of X whose ρ -distance to x is less than ε . The symbol $\pi_{\mu}X$ will be used to denote the completion of μX . Let X be a topological space. If A_1, \dots, A_n are subsets of X and $\xi(X)$ is a collection of non-empty subsets of X , then $\langle A_1, \dots, A_n \rangle^{\xi}$ is the family of all members B of $\xi(X)$ such that 1) $B \subset A_1 \cup \dots \cup A_n$ and 2) $B \cap A_i \neq \emptyset$ for each $i \in \{1, \dots, n\}$. The set $\xi(X)$ equipped with the topology generated by the basic open sets $\langle V_1, \dots, V_n \rangle$, where V_1, \dots, V_n are open subsets of X , is called a Vietoris hyperspace. The collection of all compact (resp. closed) nonempty subsets of X will be denoted by $K(X)$ (resp. $H(X)$). If ρX is a pseudometric space, then $K(X)$ is pseudometrizable by the Hausdorff pseudometric $\hat{\rho}$ defined by setting $\hat{\rho}(A_1, A_2) < \varepsilon$ iff $A_1 \subset B_{\rho, \varepsilon}(A_2)$ and $A_2 \subset B_{\rho, \varepsilon}(A_1)$. If μX is a uniform space and $\mathcal{U} \in \mu$, the entourage $\tilde{\mathcal{U}}$ is defined by setting $(A_1, A_2) \in \tilde{\mathcal{U}}$ iff $A_1 \subset \text{St}(A_2, \mathcal{U})$ and $A_2 \subset \text{St}(A_1, \mathcal{U})$. (Then the subsets A_1 and A_2 are said to be near of order \mathcal{U} .) The family $\xi(X)$ equipped with this entourage uniformity will be denoted by $\xi(\mu X)$ and the uniformity of the space will be denoted by ξ_{μ} . It was proved in [9] that K_{μ} is always compatible with the Vietoris topology of $K(X)$, on the other hand, it was also shown that the corresponding statement does not hold for H_{μ} . (See also [23].)

It is helpful to introduce an explicit covering uniformity for $K(\mu X)$. Let $\mathcal{U} \in \mu$ and write $\langle \mathcal{U} \rangle$ for the collection of all sets $\langle U_1, \dots, U_n \rangle^K$, where U_1, \dots, U_n are

elements of \mathcal{U} . Then (see [11], Lemma 1.4) the family $\langle \mathcal{U} \rangle$ is a uniform cover of $K(\mu X)$ and we shall write $\langle \mu \rangle$ for the uniformity (= $K\mu$) generated by the base $\{\langle \mathcal{U} \rangle: \mathcal{U} \in \mu\}$.

3. Some Examples of Covering Properties

We shall present some uniform covering properties preserved or not preserved in uniform hyperspaces. A uniform space is called point-finite (resp. star-finite) if it has a basis for its uniform coverings consisting of point-finite (resp. star-finite) covers.

Proposition 3.1. *If μX is point-finite (resp. star-finite), then so is $K(\mu X)$.*

Proof. For point-finiteness, suppose that μX is point-finite and let \mathcal{U} be a uniform cover of $K(\mu X)$. Then there is a uniform cover \mathcal{V} of μX such that $\langle \mathcal{V} \rangle$ refines \mathcal{U} . Since μX is point-finite, we can assume that \mathcal{V} is a point-finite cover and in fact we may assume that there is a uniform cover \mathcal{W} of μX such that each member of \mathcal{W} meets only finitely many elements of \mathcal{V} . (Just use the fact that a uniform cover has a strict uniform shrinking, see [8], IV 19.) To show that $\langle \mathcal{V} \rangle$ is point-finite, let C be a compact subset of X . Then there is a finite family \mathcal{W}' of elements of \mathcal{W} such that $C \subset \cup(\mathcal{W}')$. Now if $V_1, \dots, V_n \in \mathcal{V}$ and $C \in \langle V_1, \dots, V_n \rangle$, then C meets each V_i and thus for each $i \in \{1, \dots, n\}$ there is an element W_i of \mathcal{W}' with $W_i \cap V_i \neq \emptyset$. But each W_i meets only finitely many elements of \mathcal{V} and hence C belongs to not more than finitely many elements of $\langle \mathcal{V} \rangle$.

For star-finiteness, suppose that μX is star-finite and let \mathcal{U} be a uniform cover of $K(\mu X)$. Let \mathcal{V} be a

star-finite uniform cover of μX such that $\langle \mathcal{V} \rangle$ refines \mathcal{U} . If $V_1, \dots, V_m, W_1, \dots, W_n$ are elements of \mathcal{V} and $\langle V_1, \dots, V_m \rangle$ meets $\langle W_1, \dots, W_n \rangle$, then there is a point $C \in K(X)$ such that

$$C \subset \left(\bigcup_{i=1}^m V_i \right) \cap \left(\bigcup_{j=1}^n W_j \right) \text{ and}$$

$C \cap V_i \neq \emptyset \neq C \cap W_j$ for all i and j . Thus for each $i \in \{1, \dots, m\}$ there is a $j_i \in \{1, \dots, n\}$ with $V_i \cap W_{j_i} \neq \emptyset$ and consequently the number of all $\langle W_1, \dots, W_n \rangle$ meeting $\langle V_1, \dots, V_m \rangle$ is finite.

On the other hand, the hyperspace functor H preserves neither point-finite nor star-finite spaces. To show that H does not preserve these properties, we need a result due to J. Pelant [14]. For a cardinal number κ , let $l_\infty(\kappa)$ be the set of all bounded real-valued functions on κ with the supremum norm. Pelant proved that if κ is uncountable, then $l_\infty(\kappa)$ does not have a point-finite basis for its natural norm-induced uniformity. Let $D(\kappa)$ be a uniformly discrete space of cardinality κ . We shall write $D(\kappa) = \{d_\alpha : \alpha < \kappa\}$ and we equip $D(\kappa)$ with the metric σ for which $\sigma(d_\alpha, d_\beta) = 1$ whenever $\alpha \neq \beta, \alpha, \beta < \kappa$. Define a map $\phi: l_\infty(\omega_1) \rightarrow H(D(\omega_1) \times \mathbb{R})$ by setting $\phi(f) = \{(d_\alpha, f\alpha) : \alpha < \omega_1\}$ for each bounded function f on ω_1 . Note that ϕ is a uniform embedding and hence $H(D(\omega_1) \times \mathbb{R})$ is not point-finite even though $D(\omega_1) \times \mathbb{R}$ is star-finite.

A uniform space has the l_1 -property--introduced by Zdeněk Frolík in [2]--if for each uniform cover of the space there is a subordinated partition of unity such that the finite sums of the elements of the partition form a

uniformly equicontinuous family. It is known that the l_1 -property is productive. However, the functor K does not preserve the l_1 -property. To see this, we shall apply an example given by M. Zahradník [22]. For each positive integer n , let I^n denote the Euclidean n -cube with its standard maximum metric. Zahradník proved that the metric sum $\Sigma\{I^n\}$ does not have the l_1 -property. If μX is a uniform space, let $C(\mu X)$ denote the uniform hyperspace consisting of all nonempty subcontinua of X , considered as a uniform subspace of $K(\mu X)$. Finite-dimensional uniform spaces have the l_1 -property; especially the space $D(\omega) \times I^2$ has this property. Now define a map

$$\phi: \Sigma\{I^n: n < \omega\} \longrightarrow C(D(\omega) \times I^2)$$

by mapping the point (x_1, \dots, x_n) of I^n to the path of $\{n\} \times I^2$ composed of the segments $[i/(n+1), (i+1)/(n+1)] \times \{x_i\}$ and $\{(i+1)/(n+1)\} \times J(x_i, x_{i+1})$, where $J(x_i, x_{i+1})$ denotes the line segment from x_i to x_{i+1} . It is easy to check that ϕ is a uniform embedding and thus $C(D(\omega) \times I^2)$ --and a fortiori $K(D(\omega) \times I^2)$ --does not have the l_1 -property that is hereditary. Also note that the above example shows (this is easy, anyhow) that productive properties such as distality (every uniform cover has a finite-dimensional uniform refinement) and the Euclidean covering property (every uniform cover can be realized in some R^n) are not preserved by K .

Many statements that hold for countable powers of spaces are valid also for hyperspaces. However, the above examples show that one cannot always compare countable powers of uniform spaces with their hyperspaces of compact subsets.

Here is another example of a productive uniform property which is not preserved under K . A uniform space μX has a σ -disjoint base if each uniform cover of μX has a uniform refinement of the form $\cup\{V_n : n < \omega\}$, where the families V_n are disjoint collections of subsets of X . The property of having a σ -disjoint base is productive. On the other hand, σ -disjoint uniform covers have point-finite uniform refinements, see [16], Proposition 2.3.i. Pelant gave in [15] an example of a point-finite space having no σ -disjoint base. Let κ be an infinite cardinal and let $F(\kappa)$ denote the set of all maps f of κ into the unit interval I such that there exist only finitely many elements $\beta < \kappa$ for which $f(\beta) \neq 0$. Consider $F(\kappa)$ as a uniform subspace of $l_\infty(\kappa)$. Pelant proved that for each cardinal λ there is a cardinal $\kappa(\lambda)$ such that $F(\kappa(\lambda))$ does not have a λ -disjoint base. Given a cardinal number α , let $H(\alpha)$ --the uniform quotient space $D_\alpha \times I/D_\alpha \times \{0\}$ --denote the hedgehog with α spines and the standard geodesic metric. Define a $\phi: F(\kappa(\omega)) \rightarrow K(H(\kappa(\omega)))$ by setting

$$\phi(f) = \{p\} \cup \{(d_\beta, f(\beta)) : \beta < \kappa(\omega), f(\beta) \neq 0\},$$

where p is the "base point" of $H(\kappa(\omega))$. Let ρ be the geodesic metric of $H(\kappa(\omega))$. Then it is not difficult to see that for each pair f, g of elements of $F(\kappa(\omega))$, we have $\|f - g\|_\infty = \hat{\rho}(\phi(f), \phi(g))$. Thus, ϕ is an isometric embedding and hence $K(H(\kappa(\omega)))$ does not have a σ -disjoint base. However, $H(\kappa(\omega))$ is clearly point-finite and so is $K(H(\kappa(\omega)))$, by Proposition 2.1.

A uniform space μX is called uniformly connected, if for every uniform cover $\mathcal{U} \in \mu$ and each pair p, q of points

of X there is a simple chain of elements of \mathcal{U} connecting p and q . The hyperspace functor K preserves uniformly connected spaces. To see what is the situation with H , we need the concept of chainability. We say that μX is chainable (resp. 1-chainable) if for each uniform cover $\mathcal{U} \in \mu$ there exist a finite (resp. one-point) subset E of X and a natural number n such that $X = \text{St}^n(E, \mathcal{U})$. (For these concepts, see M. Atsuji, Pacific Math. Journal 8, 1958, and J. Hejzman, Czech. Math. Journal, 9 (84), 1959.) Chainability and 1-chainability are properties preserved by both K and H . It is easy to see that a space is 1-chainable iff it is chainable and uniformly connected. Similarly, it is easy to see that $H(\mu X)$ is uniformly connected iff μX is 1-chainable.

4. Uniform Paracompactness

A uniform space is called uniformly paracompact [19] (resp. uniformly para-Lindelöf) if every open cover of the space has a uniformly locally finite (resp. uniformly locally countable) open refinement. A topological space X is C -scattered [21] if every nonempty closed subspace F of X contains a point with a compact neighborhood in F . A generalization of this concept is useful here. Call a space X C_κ -scattered, if every nonempty closed subspace F of X contains a point with a κ -compact neighborhood in F . Recall that a space is κ -compact if every open cover of the space has a subcover with fewer than κ elements. It can be shown that a uniformly paracompact (resp. uniformly para-Lindelöf) metric space is C -scattered (resp.

$C(\omega_1)$ -scattered). (See [6], Lemma 4.2.5 and Theorem 3.1.4.)

Lemma 4.1. Let X be a metrizable space and let κ be an infinite cardinal such that either $\kappa = \omega$ or $\text{cf}(\kappa) > \omega$. Then $K(X)$ is C_κ -scattered if, and only if, X is locally κ -compact.

Proof. Suppose that X is locally κ -compact. If $\kappa = \omega$, then it follows from Proposition 4.4 of [9] that $K(X)$ is locally κ -compact. If $\text{cf}(\kappa) > \omega$, then each point x of X has an open neighborhood V_x containing a dense subset S_x of cardinality less than κ , since X is metrizable. Let C be a nonempty compact subset of X . There is a finite set $\{x_1, \dots, x_n\}$ of points of C such that C is contained in the union of the sets $V_{x(1)}, \dots, V_{x(n)}$. Now $\langle V_{x(1)}, \dots, V_{x(n)} \rangle$ is a neighborhood of C containing a dense subset

$$F(S_{x(1)} \cup \dots \cup S_{x(n)}) \cap \langle V_{x(1)}, \dots, V_{x(n)} \rangle$$

of cardinality less than κ , where we write $F(S)$ for the set of all nonempty finite subsets of a set S . But then $\langle V_{x(1)}, \dots, V_{x(n)} \rangle$ is a κ -compact neighborhood of C .

For necessity, let X be any regular space such that $K(X)$ is C_κ -scattered. We shall prove that X is locally κ -compact. Let p be an arbitrary point of X and let \mathcal{C} be the collection of all compact subsets of X containing p . Note that \mathcal{C} is a closed subspace of $K(X)$. Hence, by the assumption there exist an element C_0 of \mathcal{C} and a neighborhood \mathcal{U} of C_0 in $K(X)$ such that $\mathcal{U} \cap \mathcal{C}$ is κ -compact. As X is regular, $K(X)$ is regular by Theorem 4.9.10 in [9]. Thus, there exist open subsets G_1, \dots, G_n of X such that

$$C_0 \in \langle G_1, \dots, G_n \rangle \cap \mathcal{C} \subset \overline{\langle G_1, \dots, G_n \rangle} \cap \mathcal{C} \subset \mathcal{U}.$$

Now $\langle \overline{G_1, \dots, G_n} \rangle = \langle \overline{G_1}, \dots, \overline{G_n} \rangle$ by Lemma 2.3 in [9]. We shall show that $\overline{G_1} \cup \dots \cup \overline{G_n}$ is κ -compact. Indeed, let \mathcal{H} be a family of open subsets of X that cover $\overline{G_1} \cup \dots \cup \overline{G_n}$. Then

$$\{ \langle H_1, \dots, H_m \rangle : H_1, \dots, H_m \in \mathcal{H} \}$$

is a family of open subsets of $K(X)$ that cover $\langle \overline{G_1}, \dots, \overline{G_n} \rangle$ and by our assumption there is a subfamily $\mathcal{H}' \subset \mathcal{H}$ such that $|\mathcal{H}'| < \kappa$ and

$$\langle \overline{G_1}, \dots, \overline{G_n} \rangle \cap \mathcal{C} \subset \cup \{ \langle H_1, \dots, H_m \rangle : H_1, \dots, H_m \in \mathcal{H}' \}.$$

To show that $\overline{G_1} \cup \dots \cup \overline{G_n} \subset \cup(\mathcal{H}')$, let x be a point of $\overline{G_1} \cup \dots \cup \overline{G_n}$ and for each i choose a point $y_i \in G_i$. Then defining $C_x = \{p\} \cup \{y_1, \dots, y_n\} \cup \{x\}$ we see that $C_x \in \langle \overline{G_1}, \dots, \overline{G_n} \rangle \cap \mathcal{C}$ and that consequently there exist $H_1, \dots, H_m \in \mathcal{H}'$ such that $C_x \in \langle H_1, \dots, H_m \rangle$. Hence C_x is contained in the union of the sets H_1, \dots, H_m . Thus, there is an $H \in \mathcal{H}'$ with $x \in H$. It follows that $\overline{G_1} \cup \dots \cup \overline{G_n}$ is κ -compact and thus one of the sets $\overline{G_1}, \dots, \overline{G_n}$ is a κ -compact neighborhood of x .

Remark. If κ is an uncountable cardinal with $cf(\kappa) = \omega$, then there is a κ -compact metrizable space X such that $K(X)$ is not C_κ -scattered. Indeed, let $\{\lambda_n\}$ be a sequence of cardinals smaller than κ such that $\kappa = \sup\{\lambda_n\}$. Let $Y_n = \{(\alpha, n) : \alpha < \lambda_n\}$ and let $Z = \cup\{Y_n : n < \omega\} \cup \{\omega\}$ and define a metric d on Z by setting $d((\alpha, n), (\beta, n)) = 1/n$, $d((\alpha, m), (\beta, n)) = 2/m$ whenever $m < n$ and $d((\alpha, n), \omega) = 2/n$ for all $n < \omega$. Denote the corresponding metrizable space by X . Then X is κ -compact but $K(X)$ is not C_κ -scattered. In fact, let \mathcal{C} be the collection of all compact subsets of X containing the point ω . Then \mathcal{C} is a closed subspace of

$K(X)$. Suppose that $C \in \mathcal{C}$ and let \mathcal{U} be a neighborhood of C in \mathcal{C} . It is not difficult to see that there is an $n < \omega$ such that \mathcal{U} contains a closed subspace homeomorphic to a discrete sum of the spaces Y_i , $i > n$. However, this sum is not κ -compact just because $\text{cf}(\kappa) = \omega$. Hence, \mathcal{U} is not κ -compact, either.

Another Remark. In general, the hyperspace $K(X)$ of a locally κ -compact nonmetrizable space need not be locally κ -compact. A. Okuyama provided in [13] a cosmic space X such that $K(X)$ is not paracompact. Note that X^ω is hereditarily Lindelöf. However, $K(X^\omega)$ is not locally Lindelöf. To see this, observe that the map $K(X)^\omega \rightarrow K(X^\omega)$ sending the point (C_1, C_2, C_3, \dots) to the point $C_1 \times C_2 \times C_3 \times \dots$ is a closed embedding.

Corollary 4.2. Let ρX be a metric space. Then $K(\rho X)$ is uniformly paracompact (resp. uniformly para-Lindelöf) if, and only if, ρX is uniformly locally compact (resp. uniformly locally Lindelöf).

Proof. It is enough to note that if \mathcal{U} is a uniform cover of ρX by compact (resp. Lindelöf) sets, then $\langle \mathcal{U} \rangle$ is a uniform cover of $K(\rho X)$ by sets of the respective type.

5. Metric-Completeness in Uniform Hyperspaces

In this section we shall consider metric-completeness in uniform hyperspaces by using uniform inverse limits. It is a well-known and useful fact that the hyperspace functor K and inverse limits commute. This property was used by e.g. J. Segal in [20] and P. Zenor in [24]. For a

complete proof, see [12], p. 171. We shall show that the uniform hyperspace functor K commutes with uniform inverse limits. The proof procedure is typical but makes an essential use of Morita's covers defined in Introduction and hence we shall give a sketch. Recall that a uniform inverse system is an inverse system $\{\mu_\alpha X_\alpha, f_{\alpha\beta}, \Lambda\}$ consisting of uniform spaces $\mu_\alpha X_\alpha$, uniformly continuous bonding maps $f_{\alpha\beta}: \mu_\beta X_\beta \rightarrow \mu_\alpha X_\alpha$ and a directed set Λ . Further, recall that K has the following functorial property: if $f: \mu X \rightarrow \nu Y$ is a uniformly continuous map, then the map $K(f): K(\mu X) \rightarrow K(\nu Y)$ defined by $K(f)(C) = \{f(x): x \in C\}$ is uniformly continuous.

Theorem 5.1. Let $S = \{\mu_\alpha X_\alpha, f_{\alpha\beta}, \Lambda\}$ be a uniform inverse system. Then $\varprojlim_{\alpha} \{K(\mu_\alpha X_\alpha), K(f_{\alpha\beta}), \Lambda\}$ is uniformly isomorphic to $K(\varprojlim_{\alpha} S)$.

Proof. Write $K(S) = \{K(\mu_\alpha X_\alpha), K(f_{\alpha\beta}), \Lambda\}$ and let $\pi_\alpha: \varprojlim_{\alpha} S \rightarrow \mu_\alpha X_\alpha$ and $pr_\alpha: \varprojlim_{\alpha} K(S) \rightarrow K(\mu_\alpha X_\alpha)$ be the canonical projections. Given a compact subset C of $\varprojlim_{\alpha} S$, let $\phi(C)$ be the point of $\varprojlim_{\alpha} K(S)$ for which $pr_\alpha \phi(C) = \pi_\alpha[C]$ whenever $\alpha \in \Lambda$. It is well known that ϕ is a homeomorphism. Moreover, ϕ is uniformly continuous, since $pr_\alpha \phi = K(\pi_\alpha)$ is uniformly continuous for each $\alpha \in \Lambda$. To show that ϕ^{-1} is uniformly continuous, we make the following observation.

Observation. If $\alpha \in \Lambda$ and $W_1, \dots, W_n \subset X_\alpha$, then $\phi^{-1} pr_\alpha^{-1}[\langle W_1, \dots, W_n \rangle] = \langle \pi_\alpha^{-1}[W_1], \dots, \pi_\alpha^{-1}[W_n] \rangle$.

Let \mathcal{U} be a uniform cover of $K(\varprojlim_{\alpha} S)$. Then there is a uniform cover \mathcal{V} of $\varprojlim_{\alpha} S$ such that $\langle \mathcal{V} \rangle < \mathcal{U}$. By a

fundamental property of uniform inverse systems--see [8], IV.3.1--there is an $\alpha \in \Lambda$ and a uniform cover $\mathcal{W} \in \mu_\alpha$ such that $\mathcal{G} = \pi_\alpha^{-1}(\mathcal{W})$ refines \mathcal{V} and consequently $\langle \mathcal{G} \rangle$ refines \mathcal{U} . Let W_1, \dots, W_n be elements of \mathcal{W} and let $U \in \mathcal{U}$ be such that $\langle \pi_\alpha^{-1}[W_1], \dots, \pi_\alpha^{-1}[W_n] \rangle \subset U$. By our observation

$$\text{pr}_\alpha^{-1}[\langle W_1, \dots, W_n \rangle] = \phi[\langle \pi_\alpha^{-1}[W_1], \dots, \pi_\alpha^{-1}[W_n] \rangle] \subset \phi[U]$$

and hence $\mathcal{H} = \text{pr}_\alpha^{-1}(\langle \mathcal{W} \rangle) \subset \{\phi[U] : U \in \mathcal{U}\}$, which shows that the latter is a uniform cover since \mathcal{H} is a uniform cover of $\varprojlim K(S)$.

We stated Theorem 5.1 for the following application that will be useful in the context of metric-fine spaces. A uniform space μX is called metric-complete if every μ -Cauchy filter with the countable intersection property converges. Such uniform spaces were considered independently by Morita [10]--who called them weakly complete--and Rice [17]. Following the terminology of [17], the smallest metric-complete uniform subspace of $\pi\mu X$ containing μX will be denoted by $d\mu X$. Then $d\mu X = \{p \in \pi\mu X : \text{each } G_\delta\text{-set containing } p \text{ meets } X\}$. Thus, $d\mu X$ is the G_δ -closure of μX in its completion. The metric-completion can be characterized using inverse limits as in [10] and [16]. Let $\mathcal{P} = \{\rho_\alpha : \alpha \in \Lambda\}$ be the set of all uniformly continuous pseudometrics on μX . For each $\alpha \in \Lambda$ let $[\rho_\alpha X]$ be the metric quotient space assigned to the pseudometric space $\rho_\alpha X$ and let $\pi_\alpha : \rho_\alpha \rightarrow [\rho_\alpha X]$ be the quotient map. Define a relation $<$ on Λ by setting $\alpha < \beta$ if the identity map $i_{\alpha\beta} : \rho_\beta X \rightarrow \rho_\alpha X$ is uniformly continuous. If $\alpha < \beta$, then there is a natural uniformly continuous surjective map $f_{\alpha\beta} = [i_{\alpha\beta}]$ of $[\rho_\beta X]$

onto $[\rho_\alpha X]$ such that $f_{\alpha\beta}\pi_\beta = \pi_\alpha$. We obtain an inverse system $\{[\rho_\alpha X], f_{\alpha\beta}, \Lambda\}$ of metric spaces for which $\mu X = \varprojlim [\rho_\alpha X]$. We shall apply Theorem 5.1 to show that $Kd = dK$. To that end we need two lemmas.

Lemma 5.2. *Let μX be a uniform space and let ρ_1, \dots, ρ_m be uniformly continuous pseudometrics on $K(\mu X)$. Then there exists a uniformly continuous pseudometric σ on μX such that*

$$(\rho_1 \vee \dots \vee \rho_m) \wedge \underline{1} \leq \hat{\sigma}.$$

Proof. For each n , and each $i \in \{1, \dots, m\}$ there exists a uniform cover $\mathcal{U}_{n,i} \in \mu$ such that if $C_1, C_2 \in K(X)$ are near of order $\mathcal{U}_{n,i}$, then $\rho_i(C_1, C_2) < 2^{-n}$. Let $\mathcal{U}_n = \mathcal{U}_{n,1} \wedge \dots \wedge \mathcal{U}_{n,m}$. Obviously we can assume that the covers \mathcal{U}_n form a normal sequence $\dots \prec^* \mathcal{U}_n \prec^* \mathcal{U}_{n+1} \dots$. By the Alexandroff-Urysohn Metrization theorem there exists a uniformly continuous pseudometric δ on μX such that

$$St(x, \mathcal{U}_{n+3}) \subset B_{\delta, 2^{-n-1}}(x) \subset St(x, \mathcal{U}_n)$$

for all n and $x \in X$. Put $\sigma = 4\delta$. Suppose that C_1, C_2 are nonempty compact subsets of X and that $0 < \hat{\sigma}(C_1, C_2) < 1$. Choose an n such that $2^{-n-1} \leq \hat{\sigma}(C_1, C_2) < 2^{-n}$. It follows that

$$C_2 \subset B_{\delta, 2^{-n-2}}(C_1) \subset St(C_1, \mathcal{U}_{n+1})$$

and similarly $C_1 \subset St(C_2, \mathcal{U}_{n+1})$. Consequently the sets C_1 and C_2 are near of order \mathcal{U}_{n+1} and thus for each $i \in \{1, \dots, m\}$, $\rho_i(C_1, C_2) < 2^{-n-1} \leq \hat{\sigma}(C_1, C_2)$. On the other hand, if $\hat{\sigma}(C_1, C_2) = 0$, then $C_1 \subset St(C_2, \mathcal{U}_n)$ and $C_2 \subset St(C_1, \mathcal{U}_n)$ for all n and thus for each $i \in \{1, \dots, m\}$, we have $\rho_i(C_1, C_2) = 0$.

If ρX and σY are pseudometric spaces and $f: \rho X \rightarrow \sigma Y$ is a map, define the natural map $[f]: [\rho X] \rightarrow [\sigma Y]$ by setting $[f](x) = [f(x)]$.

Lemma 5.3. Let μX be a uniform space and let ρ be a uniformly continuous pseudometric on μX . Then $[\hat{\rho}K(X)]$ is isometric to $K([\rho X])$. If σY is a pseudometric space and $f: \rho X \rightarrow \sigma Y$ is a map, then $[K(f)] = K([f])$.

Proof. Let $g: \rho X \rightarrow [\rho X]$ and $h: \hat{\rho}K(X) \rightarrow [\hat{\rho}K(X)]$ be the quotient maps. If $C \in K([\rho X])$, then it is not difficult to see that $g^{-1}[C]$ is a compact subset of ρX . Thus, we can define a map ϕ of $K([\rho X])$ into $[\hat{\rho}K(X)]$ by setting $\phi(C) = h[g^{-1}[C]]$. It is a straightforward exercise to verify that ϕ is an isometry. By the same token the proof of the second assertion is routine.

By Lemma 5.3, we consider $[\hat{\rho}K(X)]$ and $K([\rho X])$ as identical spaces.

Theorem 5.4. Let μX be a uniform space. Then $dK(\mu X) = K(d\mu X)$.

Proof. Let $\mathcal{P} = \{\rho_\alpha: \alpha \in \Lambda\}$ be the set of all uniformly continuous pseudometrics on $K(\mu X)$ and let $\mathcal{P}^* = \{\sigma_\alpha: \alpha \in \Lambda^*\}$ be the set of all uniformly continuous pseudometrics on μX indexed so that $\Lambda^* \subset \Lambda$ and for each $\alpha \in \Lambda^*$, $\rho_\alpha = \hat{\sigma}_\alpha$. Let $\mathcal{S} = \{[\rho_\alpha K(X)], f_{\alpha\beta}, \Lambda\}$, $\mathcal{S}^* = \{[\sigma_\alpha X], g_{\alpha\beta}, \Lambda^*\}$ be the natural inverse systems associated with \mathcal{P} and \mathcal{P}^* , respectively, such that $dK(\mu X) = \varprojlim \mathcal{S}$ and $d\mu X = \varprojlim \mathcal{S}^*$. Further, let $\hat{\mathcal{S}} = \{[\hat{\sigma}_\alpha K(X)], f_{\alpha\beta}, \Lambda^*\}$. By Lemma 5.3 we have $f_{\alpha\beta} = [j_{\alpha\beta}] = [K(i_{\alpha\beta})] = K([i_{\alpha\beta}]) = K(g_{\alpha\beta})$, where $j_{\alpha\beta}: \hat{\sigma}_\beta K(X) \rightarrow \hat{\sigma}_\alpha K(X)$

and $i_{\alpha\beta}: \sigma_\beta X \rightarrow \sigma_\alpha X$ are identity maps. Given $\alpha_1, \dots, \alpha_n \in \Lambda$, Lemma 5.2 shows that there exists a $\beta \in \Lambda^*$ with

$(\rho_{\alpha_1} \vee \dots \vee \rho_{\alpha_n}) \wedge \underline{1} \leq \hat{\sigma}_\beta$; as a consequence, the identity maps $j_{\alpha_i\beta}$ are uniformly continuous. Thus, $\alpha_i < \beta$ for

every $i \in \{1, \dots, n\}$. Therefore, \hat{S} is a cofinal part of S . By a fundamental theorem on cofinal parts of inverse systems--see e.g. [8], IV.35--the limits $\varprojlim S$ and $\varprojlim \hat{S}$ are uniformly isomorphic, written $\varprojlim S \cong \varprojlim \hat{S}$. Using Theorem 5.1 and Lemma 5.3, we obtain

$$\begin{aligned} dK(\mu X) &= \varprojlim S \cong \varprojlim \hat{S} = \varprojlim \{[\hat{\sigma}_\alpha K(X)], f_{\alpha\beta}, \Lambda^*\} \\ &\cong \varprojlim \{K([\sigma_\alpha X]), K(g_{\alpha\beta}), \Lambda^*\} \\ &= K(\varprojlim S^*) = K(d\mu X). \end{aligned}$$

Remark. Morita's theorem [11] that $K\pi = \pi K$ can be obtained as a corollary to Theorem 5.1. Indeed, let μX be a uniform space. Then μX is the inverse limit of a system $\{\rho_\alpha X\}$ of pseudometric spaces with the underlying set X and the identity maps as bonding maps. In this case it is easy to see that π commutes with the limit operation. On the other hand, it follows directly from II.48 of [8] that H and π --and a fortiori K and π --commute on pseudometric spaces. Thus, $\pi K(\mu X) = \pi K(\varprojlim \rho_\alpha X) = \pi \varprojlim K(\rho_\alpha X) \simeq \varprojlim \pi K(\rho_\alpha X) \simeq \varprojlim K(\pi \rho_\alpha X) \simeq K(\varprojlim \pi \rho_\alpha X) \simeq K(\pi \varprojlim \rho_\alpha X) \simeq K(\pi \mu X)$. (See also [1].)

6. Locally Fine Spaces

A uniform space is called locally fine [4] if every uniformly locally uniform cover of the space is uniform. This means that μX is locally fine provided that every cover

of X of the form $\{U_i \cap V_j^i\}$, where $\{U_i\}$ and $\{V_j^i\}$ are uniform covers of μX , is a uniform cover. Given a uniform space μX , the discreteness character $\delta(\mu X)$ is the least cardinal κ such that $|D| < \kappa$ for every uniformly discrete subset of μX . The following lemma is essentially known as a part of mathematical folklore.

Lemma 6.1. *Let μX be a uniform space. Then*

i) $\delta(K(\mu X)) = \delta(\mu X)$ and

ii) $\delta(H(\mu X)) = \sup\{2^\kappa : \kappa < \delta(\mu X)\}$

whenever $\delta(\mu X)$ is infinite.

Proof. The proof showing that uniform hyperspaces of precompact spaces are precompact can readily be modified to prove the above lemma.

Let κ be an infinite cardinal. A uniform space μX admits the cardinal κ if for each family $\{\mathcal{U}_i : i \in I\}$ of uniform covers of μX with $|I| < \kappa$ there is a common uniform refinement. ([8], p. 733.)

Theorem 6.2. *Let μX be a uniform space. Then $K(\mu X)$ (resp. $H(\mu X)$) is locally fine if, and only if, μX admits every cardinal $\kappa < \delta(\mu X)$ (resp. admits 2^κ for every $\kappa < \delta(\mu X)$).*

Proof. We shall prove the result for $H(\mu X)$ because the proof for $K(\mu X)$ is the same modulo simplifying changes. To prove necessity, let $\kappa < \delta(\mu X)$ and let D be a subset of X , uniformly discrete relative to a uniform cover \mathcal{U} of μX , such that $|D| = \kappa$ and let $\{\mathcal{V}_\alpha : \alpha < 2^\kappa\}$ be a family of uniform covers of μX . Write $D = D_1 \cup D_2$ where

$|D_1| = |D_2| = \kappa$ and $D_1 \cap D_2 = \emptyset$; and for $i = 1, 2$, define $\mathcal{D}_i = P(D_i) - \{\emptyset\}$, where $P(D_i)$ is the power set of D_i . Let $\mathcal{W} \in \mu$ be such that $\mathcal{W} <^{***} \mathcal{U}$ and let $Y_i = X - \text{St}(D_i, \mathcal{W})$.

For $i = 1, 2$, write $\mathcal{D}_i = \{D_{i,\alpha} : \alpha < 2^K\}$ and define $A_{i,\alpha} = \{D_{i,\alpha} \cup \{x\} : x \in Y_i\}$. Then $\alpha, \beta < 2^K, \alpha \neq \beta$ imply that the families $A_{i,\alpha}$ and $A_{i,\beta}$ are $\tilde{\mathcal{W}}$ -disjoint as subsets of $H(X)$. (Note that the elements of every $A_{i,\gamma}$ are closed subsets of X since the families D_i are *uniformly discrete*.)

As $H(\mu X)$ is locally fine, so is the subspace $A_i = \cup\{A_{i,\alpha} : \alpha < 2^K\}$. Let $\alpha < 2^K, V \in \mathcal{V}_\alpha$ and put $V_{i,\alpha} = \{D_{i,\alpha} \cup \{x\} : x \in V \cap Y_i\}$. Then $\mathcal{V}_i = \{V_{i,\alpha} : V \in \mathcal{V}_\alpha, \alpha < 2^K\}$ is a uniformly locally uniform cover of A_i --since $\{A_{i,\alpha} : \alpha < 2^K\}$ is a uniform cover of A_i . By the assumption there exists a uniform cover $\mathcal{G}_i < \mathcal{W}$ of μX such that if $F_1 \in A_i$, then there is a $V \in \mathcal{V}_i$ with the following property: if $F_2 \in A_i$ and the sets F_1 and F_2 are near of order \mathcal{G}_i , then $F_2 \in V$. We shall show that $\mathcal{G}_i|_{Y_i} < \mathcal{V}_\alpha|_{Y_i}$. Indeed, let $x \in Y_i$. Then $(D_{i,\alpha} \cup \{x\}) \in A_i$ and thus we can find a $V \in \mathcal{V}_\alpha$ such that if $F \in A_i$ is \mathcal{G}_i -near to $D_{i,\alpha} \cup \{x\}$, then $F \in V_{i,\alpha}$. Now $y \in \text{St}(x, \mathcal{G}_i) \cap Y_i$ implies that the sets $D_{i,\alpha} \cup \{x\}$ and $D_{i,\alpha} \cup \{y\}$ are near of order \mathcal{G}_i : therefore $(D_{i,\alpha} \cup \{y\}) \in V_{i,\alpha}$ and consequently $y \in V \cap Y_i$. Accordingly, we have $\text{St}(x, \mathcal{G}_i) \cap Y_i \subset V \cap Y_i$. Now $\{Y_1, Y_2\}$ is a uniform cover of μX and it follows that the covers \mathcal{V}_α , where $\alpha < 2^K$, have a common uniform refinement $\mathcal{G}_1 \wedge \mathcal{G}_2 \wedge \{Y_1, Y_2\}$ in μ .

For sufficiency, let $\{V_i \cap W_j^i\}$ be a typical uniformly locally uniform cover of $H(\mu X)$, where the families $\{V_i : i \in I\}$ and $\{W_j^i : j \in J_i\}$ are uniform covers of $H(\mu X)$.

By Lemma 6.1 we can assume that $|I| < \sup\{2^\kappa : \kappa < \delta(\mu X)\}$. But then μX admits $|I|$. For each $i \in I$ choose a uniform cover \mathcal{U}_i of μX such that $\{\tilde{\mathcal{U}}_i[F] : F \in H(X)\} < \{W_j^i\}$. Since μX admits the cardinality of I , the covers \mathcal{U}_i have a common uniform refinement \mathcal{U} . Then $\{\tilde{\mathcal{U}}[F] : F \in H(X)\} \wedge \{V_i\}$ is a uniform cover of $H(\mu X)$ that refines $\{V_i \cap W_j^i\}$. This shows that $H(\mu X)$ is locally fine.

The following corollary gives a more accurate description of the class of uniform spaces whose hyperspaces of closed nonempty subsets is locally fine.

Corollary 6.3. *Let μX be a uniform space. Then $H(\mu X)$ is locally fine if, and only if, either*

- i) μX is uniformly discrete, or
- ii) $2^\kappa < \delta(\mu X)$ for every $\kappa < \delta(\mu X)$ --i.e. $\delta(\mu X)$ is a strong limit cardinal--and μX admits each cardinal $\kappa < \delta(\mu X)$.

Proof. Suppose that $H(\mu X)$ is locally fine. By Theorem 6.2, μX admits 2^κ for all $\kappa < \delta(\mu X)$. Assume that there is a $\kappa_0 < \delta(\mu X)$ with $2^{\kappa_0} \geq \delta(\mu X)$. Then μX admits $\delta(\mu X)$ and by VII.27 of [8] either μX is uniformly discrete or contains a uniformly discrete subspace of cardinality $\delta(\mu X)$. The latter case being impossible, we conclude that either $2^\kappa < \delta(\mu X)$ for every $\kappa < \delta(\mu X)$ or μX is uniformly discrete. Sufficiency follows directly from Theorem 6.2.

Recall the definition of the beth numbers $[$: we have $[_0 = \omega$, $[_{n+1} = 2^{[n]}$ and $[_\omega = \sup\{[n : n < \omega\}$.

Corollary 6.4. If μX is a nonprecompact uniform space such that $\delta(\mu X) < [\omega$, then $H(\mu X)$ is locally fine if, and only if, μX is uniformly discrete.

Call a uniform space μX superfine if $H(\mu X)$ is fine. Corollary 6.4 shows that nonprecompact nontrivial superfine spaces are of "high" cardinality.

7. Metric-Fine Hyperspaces

A uniform space μX is called metric-fine provided that for every metric space ρM and every uniformly continuous map $f: \mu X \rightarrow \rho M$, the map $f: \mu X \rightarrow \mathcal{J}M$ into the fine space $\mathcal{J}M$ is uniformly continuous. Metric-fine spaces have been characterized in the separable case by Hager [5] and in the general case by Frolík [3] and Rice [18]. Following [3], μX is metric-fine iff the uniformity contains all covers $\{B_n \cap U_a^n\}$, where $\{B_n\}$ is a countable cover by uniform cozero-sets and each $\{U_a^n\}$ is a uniform cover. Hager proved in [5] that a precompact space is metric-fine iff the space is G_δ -dense in its Samuel compactification. A space which admits ω is metric-fine.

Theorem 7.1. Let μX be a uniform space. Then the following conditions are equivalent:

- i) $K(\mu X)$ (resp. $H(\mu X)$) is metric fine;
- ii) either μX is precompact and metric-fine or μX admits an infinite cardinal.

Proof. For the implication ii) \rightarrow i), assume that μX is precompact and metric-fine. Then μX is G_δ -dense in its Samuel compactification and therefore $d\mu X = \pi\mu X$. By

Theorem 5.4, $dK(\mu X) = K(d\mu X) = K(\pi\mu X) = \pi K(\mu X)$ and thus $K(\mu X)$ is G_δ -dense in its Samuel compactification. Consequently the precompact space $K(\mu X)$ is metric-fine. On the other hand, if μX admits an infinite cardinal, then so does $K(\mu X)$. In case $d\mu X = \pi\mu X$ and μX is precompact, we also have $\pi H(\mu X) = H(\pi\mu X) = K(\pi\mu X) = K(d\mu X) = dK(\mu X)$ and hence $\pi H(\mu X) = dH(\mu X)$ proving the claim for the hyperspace functor H .

Now suppose that $K(\mu X)$ is metric-fine. If $K(\mu X)$ is precompact, then it is G_δ -dense in its completion and it follows as above that $K(d\mu X) = K(\pi\mu X)$. Since $K(\pi\mu X)$ is complete and $d\mu X$ is uniformly isomorphic to a closed subspace of $K(\pi\mu X)$, $d\mu X$ is complete and thus $d\mu X = \pi\mu X$; i.e. μX is G_δ -dense in its Samuel compactification and thus metric-fine. The proof for H is similar. If μX is not precompact, then we can choose a uniformly discrete subset D of μX such that $|D| = \omega$. Let ρ be a uniformly continuous pseudometric on μX such that $\rho(d, d') \geq 1$ for distinct elements d, d' of D . The rest of the proof follows that of Theorem 6.2 and hence we give a sketch only. Write $D = D_1 \cup D_2$ where $|D_1| = |D_2| = \omega$ and $D_1 \cap D_2 = \emptyset$. For $i = 1, 2$, write $D_i = \{d_{i,n} : n < \omega\}$ and let $A_{i,n}$ be the collection of all $\{x, d_{i,n}\}$, where $x \in X - B_{\rho, 1/4}(D_i)$. Then $A_{i,n}$ is a subset of $K(X)$. Let $C_{i,n}$ denote the $1/8$ -neighborhood of $A_{i,n}$ relative to the Hausdorff pseudometric $\hat{\rho}$ and write

$$C_{i,n}^* = \{C \in C_{i,n} : \hat{\rho}(C, K(X) - C_{i,n}) > 1/16\},$$

$$B_i = K(X) - \bar{B}_{\hat{\rho}, 1/32}(\cup\{C_{i,n}^* : n < \omega\}).$$

For $i = 1, 2$, $\{B_i\} \cup \{C_{i,n} : n < \omega\}$ is a countable cover of $K(X)$ consisting of uniform cozero-sets of $K(\mu X)$. Let $\{\mathcal{U}_n\}$ be a sequence of uniform covers of μX . As in the proof of Theorem 6.2, we can form uniform covers $\mathcal{U}_{i,n}$ of the sets $C_{i,n}$. Since $K(\mu X)$ is metric-fine, the cover $\cup\{\mathcal{U}_{i,n} : n < \omega\} \cup \{B_i\}$ is a uniform cover and it follows that the covers \mathcal{U}_n have a common uniform refinement. Thus, μX admits an infinite cardinal.

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