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HOMOGENEOUS CURVES

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HOMOGENEOUS CURVES

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A continuum is a compact, connected metric space. A continuum M is homogeneous if for each pair of points p,q of M there is a homeomorphism h of M onto itself with h(p) = q. A curve is a one-dimensional continuum.

In 1920, Knaster and Kuratowski [25] asked if the simple closed curve is the only non-degenerate homogeneous plane continuum. In 1924 Mazurkiewicz [29] showed that the answer is yes if the continuum is locally connected. In 1951 Cohen [9] improved this by showing that if the homogeneous plane continuum is either arcwise connected or contains a simple closed curve then it is itself a simple closed curve. Cohen's result was strengthened in 1960 when Bing [1] showed that the simple closed curve is the only homogeneous plane continuum which contains an arc.

A false affirmative solution to Knaster and Kuratowski's question was announced in 1937 [33] and extended in 1944 [8] in an attempt to classify all homogeneous plane compacta. The solution was only discovered to be false in 1948 when Bing [2], and shortly thereafter Moise [30], proved that the pseudo-arc is homogeneous. The dispute remained alive when in 1953 Kapuano presented two papers [23], [24] purporting to show that the pseudo-arc is not homogeneous. Since the proofs of Bing and Moise are valid, and since the

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false affirmative solutions seem to depend on a confusion of the ideas of separating point and cut point of a continuum, the answer to Knaster and Kuratowski's question is no.

This of course leads to the problem of classifying all homogeneous plane continua. All such continua can be divided into three categories:

i) those which fail to separate the plane,

ii) those which separate the plane and are decomposable, and

iii) those which separate the plane and are indecomposable.

The point and the pseudo-arc are the only known examples of type i). The circle and the circles of pseudo-arcs are the only known examples of type ii). (The circle of pseudo-arcs is a circularly chainable planar continuum with a continuous decomposition into pseudo-arcs, such that the decomposition space is a simple closed curve. It was described by Bing and Jones in 1954 [6], and proven homogeneous and unique.) There is no known example of type iii), and this past spring Rogers [32] showed that no continua of type iii) exist.

In 1951, Jones [19] showed that any continuum of type i) must be indecomposable. At the Auburn Topology conference in 1969, he outlined a proof [20] that any indecomposable homogeneous plane continuum must be hereditarily indecomposable. In 1976 Hagopian [14] provided the details of such an argument.

Jones [21] also showed that any decomposable homogeneous plane continuum admits a continuous decomposition into mutually homeomorphic, non-separating homogeneous plane continua such that the decomposition space is a simple closed curve.

Bing [3] has shown that the pseudo-arc is the only homogeneous, non-degenerate, chainable continuum. Thus if there is another homogeneous plane continuum, there must exist one which is hereditarily indecomposable, tree-like, and non-chainable, since the only other possibility is a circle of such continua (à la the circle of pseudo-arcs).

There are known examples [17] of non-homogeneous hereditarily indecomposable, non-chainable, tree-like plane continua. However considerable restrictions can be placed on any such homogeneous examples. For example, the continua constructed by Ingram [17], as well as many earlier candidates [4], []0], [13] for homogeneous continua, have the property that every non-degenerate proper subcontinuum is a pseudo-arc. All of these continua are almost chainable, and, in fact, for homogeneous continua almost chainability is equivalent to every non-degenerate proper subcontinuum being a pseudo-arc [7]. We will prove that almost chainable homogeneous continua are chainable, and hence pseudo-arcs. Our argument does not assume that the continua are either planar or tree-like. At the Birmingham topology conference this past spring, Jim Davis announced the slightly weaker result that almost chainable homogeneous continua have span zero. (While chainable continua have span zero [26], it is still an open question whether span zero implies chainability.)

However, most of the restrictions on homogeneous, tree-like, plane continua are also applicable to any homogeneous, tree-like, hereditarily indecomposable continua regardless of whether they are planar or not. Jones' result [19] that homogeneous tree-like continua are indecomposable does not assume that the continua are planar, but his and Hagopian's result [20], [14] that homogeneous indecomposable plane continua are hereditarily indecomposable does strongly use properties of the plane. Thus for the non-planar case it is at least still conceivable that there is a homogeneous tree-like continuum which is indecomposable but not hereditarily indecomposable, though this seems quite unlikely. In fact it is still an open question whether such a continuum can contain an arc. (It is known that not all of its non-degenerate proper subcontinua can be arcs, since by Hagopian's result [15] this would make it a solenoid.)

While for many purposes, the author finds it most convenient to consider hereditarily indecomposable homogeneous continua, it would be quite interesting to see any further results on non-planar homogeneous continua (not necessarily curves) which are not hereditarily indecomposable.

For the hereditarily indecomposable case, Bing [5] has shown that any finite-dimensional homogeneous continuum M must be one-dimensional, since if M is n-dimensional there exists a point $p \in M$ such that every non-degenerate subcontinuum containing p is n-dimensional, while there also exist points of M contained in one-dimensional subcontinua. There do, however, exist hereditarily infinite-dimensional, hereditarily indecomposable continua, and a homogeneous such example has not yet been excluded.

For the case of an hereditarily indecomposable, homogeneous continuum M which is not a pseudo-arc, the result of this paper shows that M must contain proper subcontinua which are not chainable, and in fact there is a continuous decomposition of M into such continua. There is also a continuous decomposition of M into maximal chainable subcontinua (i.e. maximal pseudo-arcs) such that the decomposition space $\tilde{M}$ is also hereditarily indecomposable and homogeneous. Unless there exists a non-chainable, hereditarily indecomposable continuum with a continuous decomposition into pseudo-arcs such that the decomposition space is a pseudo-arc (which seems unlikely), then $\tilde{M}$ would be a homogeneous, hereditarily indecomposable continuum with no non-degenerate chainable subcontinua. There are known examples [18] of hereditarily indecomposable tree-like continua with no non-degenerate chainable subcontinua. However it is not known if a planar example exists, or a homogeneous one.

It follows from the result of this paper that any non-chainable, homogeneous, hereditarily indecomposable curve M cannot be k-junctioned for any k, i.e. if M is written as an inverse limit of finite graphs there cannot be a bound on the number of vertices of the graph of order greater than 2--or equivalently there is no bound on the number of junction links of one-dimensional covers of sufficiently small mesh.

In 1955, Jones [22] asked about the effect of hereditary equivalence on homogeneous continua. (A continuum is hereditarily equivalent if it is homeomorphic to each of its non-degenerate subcontinua.) The pseudo-arc is both homogeneous and hereditarily equivalent [31], and is the only known hereditarily equivalent continuum other than the arc. Cook [11] has shown that any other example must be hereditarily indecomposable, tree-like, and non-chainable. Little is known about the specific effects of hereditary equivalence and homogeneity on each other, but work in both areas suggests this is a question worth further study.

Almost Chainable Homogeneous Continua

A continuum M is almost chainable if for every $\varepsilon > 0$ there exists an open cover U of M, of mesh less than ε, such that U contains a chain $C = C(0), C(1), \cdots, C(n)$ with no $C(i)$ intersecting an element of $U - C$ for $i < n$, and every point of M being within a distance ε of a link of C. For homogeneous continua, almost chainability is equivalent to every non-degenerage proper subcontinuum being a pseudo-arc.

As in most recent work on homogeneous continua, we will make heavy use of the following result.

Theorem 1. [12], [16] *If* M *is a homogeneous continuum and* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* x *and* y *are two points of* M *within a distance* δ *of each other, then there is a homeomorphism* $h: M \to M$ *moving no point more than* ε *such that* $h(x) = y$.

The proof of the main theorem can be broken down into a sequence of four lemmas. For convenience, if U is an open cover of M as in the definition of almost chainability, U will be called an ε-*almost chain*, C will be called the *chain part of* U, C(0) will be called the *free end* of C, and C(n) the *non-free end* of C.

Lemma 1. *If* C *and* D *are the chain parts of almost chains* U *and* V, *respectively, then there exist chains* C' *and* D' *such that:*

i) $C'(i) \subset C(i)$ *and* $D'(i) \subset D(i)$ *for each* i.

ii) $C'^* \cup D'^* = C^* \cup D^*$.

iii) $C'^* \cap D'^*$ is a subset of the union of the non-free ends of C' and D'.

Proof. Since $Bd(C^*) \subset C(n)-C(n-1)$ and $Bd(D^*) \subset D(m)-D(m-1)$, $C^* \cap D^*$ is an open set whose boundary is contained in $(C(n)-C(n-1)) \cup (D(m)-D(m-1))$. If $Bd(C^* \cap D^*) \cap (C(n)-C(n-1)) = \emptyset$, then $C'(i) = C(i)$ for $0 \le i \le n$ and $D'(i) = D(i) - C^*$ for $0 \le i \le m$. If $Bd(C^* \cap D^*) \cap (D(m)-D(m-1)) = \emptyset$, then $D'(i) = D(i)$ for $0 \le i \le m$ and $C'(i) = C(i) - D^*$ for $0 \le i \le n$. If $Bd(C^* \cap D^*) \cap (C(n)-C(n-1)) \ne \emptyset$, and $Bd(C^* \cap D^*) \cap (D(m)-D(m-1)) \ne \emptyset$, then $C'(i) = C(i)$ for $0 \le i \le n$ and $D'(i) = D(i) - C^*$ if $D_i \cap Bd(C^*) = \emptyset$ and $0 \le i \le m$.

Lemma 2. *If* p *is a point of the homogeneous almost chainable continuum* M *and* $\varepsilon > 0$, *there is a cover* U *of* M *with mesh less than* ε *such that* U *consists of a finite number of chains and circular chains such that the only intersection of any two chains or circular chains is in the link containing* p.

Proof. For every point $x \in M - \{p\}$ there exists an $\varepsilon/3$-almost chain U_x covering M such that x is in the free end of U_x and the non-free end of U_x is within $\varepsilon/3$ of the point p. Let U_p be the $\varepsilon/3$ neighborhood of p. Let C_x be the chain part of U_x. Then $\{C_x^*\}_{x \in M-\{p\}} \cup \{U_p\}$ is an open cover of M and so there exist $x_1, x_2, \cdots, x_n$ such that $\{C_{x_i}^*\}_{1 \leq i \leq n} \cup \{U_p\}$ is a cover of M. By repeated applications of Lemma 1 we can modify the C_{x_i}'s such that any intersections are ones involving non-free ends of the C_{x_i}'s. Amalgamate all the non-free ends of the modified C_{x_i}'s together with U_p to form a link L of diameter less than ε containing p. The rest of M is covered by the links of the modified C_{x_i}'s and since the only intersections of the C_{x_i}'s involve non-free ends the cover consisting of L plus the links of the C_{x_i}'s not contained in L is the desired cover.

Lemma 3. *If* M *is a homogeneous almost chainable continuum and* $\varepsilon > 0$, *then there is an open cover* U *of* M *of mesh less than* ε *which is star-like (i.e. the nerve of* U *is a tree with only one branch point).*

Proof. Let δ be an Effros number of $\varepsilon/3$ as guaranteed by Theorem 1. Let V be an open cover of M of mesh less than δ satisfying the conclusion to Lemma 2, with L the junction link of V. Let $C = \{C_0, C_1, \cdots, C_n = C_0\}$ be a circular chain in V, labelled such that $C_0 = C_n = L$. Let P_0 and P_1 be pseudo-arcs in C^* such that P_0 runs through $C_0, C_1, C_2, \cdots, C_{n-1}$ (not intersecting $C_0 - C_1$) and P_1 runs through $C_n, C_{n-1}, C_{n-2}, \cdots, C_1$ (not intersecting $C_n - C_{n-1}$). Let p be a point of P_0 in C_0, and q a point of P_1 in C_n ($= C_0$). Let $h: M \to M$ be a homeomorphism of M moving no point more than $\varepsilon/3$ such that $h(p) = q$. Then $h(P_0) \subset P_1$ or $h(P_0) \supset P_1$. For some $i \geq 1$ there exists a point $r \in P_0 \cap (C_i - C_{i-1})$ such that $h(r) \in C_i \cap P_1$. Let D_0 be a chain of sets open in M covering the subcontinuum of P_0 irreducible from p to r, containing p and r in opposite end links, such that $\mathrm{Bd}(D_0) \subset C_0 \cap C_1$, refining C and of sufficiently small mesh that for each link d of D_0, h(d) is a subset of a link of $C_1, C_2, \cdots, C_n$. Let D_1 be a chain of sets open in M covering the subcontinuum of P_1 irreducible between q and h(r), containing q and h(r) in opposite end links, such that $\mathrm{Bd}(D_1) \subset C_n \cap C_{n-1}$ and D_1 refines C. Let $g_0: \{0,1,2,\cdots,m_0\} \to \{1,2,\cdots,n_0\}$ be a pattern which D_0 follows in C such that r is in the m_0-th link of D_0 and let $g_1: \{0,1,2,\cdots,m_1\} \to \{n_1, n_1+1, \cdots, n-1\}$ be a pattern which D_1 follows in C such that h(r) is in the 0-th link of D_1. Define the pattern $g: \{0,1,\cdots,m_0+m_1+1\} \to \{1,2,\cdots n\}$ by $g(i) = g_0(i)$ for $i \leq m_0$ and $g(i) = g_1(i-m_0-1)$ for $i > m_0$. By an argument similar to that for Theorem 3 of [27] there exists a chain D covering the part of M in $C_1, C_2, \cdots, C_{n-1}$ such that D follows the pattern g in C, $C_0 \cap C_1$ is in the first link of D and $C_n \cap C_{n-1}$ is in the last link of D. For $i \leq m_0$, let $g_2(i)$ be an index in the ordering $C_1, C_2, \cdots, C_{n-1}$ such that the image under h of the i-th link of D_0 is contained in $C_{g_2(i)}$. Let E be the chain formed by amalgamating D such that the k-th link of E contains the i-th link of D if $g_2(i) = k$ and $i \leq m_0$ or $g_1(i-m_0-1) = k$ and $i \geq m_0$. Then E is an ε-chain covering $C_1, C_2, \cdots, C_{n-1}$ with boundary only its end link intersecting $C_0 = C_n$. Doing this for each circular chain in V converts it to the desired ε-cover with a star-like nerve.

Lemma 4. *Every star-like homogeneous continuum* M *is chainable.*

Proof. Let $\varepsilon > 0$ and let $\delta > 0$ be an Effros number for $\varepsilon/6$ as guaranteed by Theorem 1. Let U be a star-like tree cover of M of mesh less than δ, with L being the junction link of U. Let A be the arms of U, i.e. the collection of chains in U maximal with respect to containing L as an end link. If $\alpha \in A$, let P_α be a pseudo-arc in α^*, intersecting both end links of α but intersecting L only in its intersection with the adjacent link of α and let P_α^0 be a point of P_α in L. For each pair $\alpha, \alpha' \in A$ there exists a homeomorphism $h_{(\alpha,\alpha')}$ of M onto itself, moving no point more than $\varepsilon/6$ with $h_{(\alpha,\alpha')}(P_\alpha^0) = P_{\alpha'}^0$. By composing each of these homeomorphisms with the inverse of one of them, there exists $\tilde{\alpha} \in A$ and homeomorphisms h_α, each moving no point more than $\varepsilon/3$, such that $h_\alpha(P_\alpha^0) = P_{\tilde{\alpha}}^0$ and $h_\alpha(P_\alpha) \subset P_{\tilde{\alpha}}$. Let C_α be a chain refining α, covering P_α, having no link in the part of L outside the rest of α, having one end link containing P_α^0, and of sufficiently small mesh that the image under h_α of each link of C_α is contained in a link of $\tilde{\alpha}$. Let g_α be a pattern which C_α follows in α, with $g_\alpha(0)$ being the link of α adjacent to L. By the same argument as in the proof of Theorem 2 of [27], the part of α^* contained in links other than L can be amalgamated to a chain D_α which follows the pattern g_α in α, with the part common to L and the adjacent link of α contained in the 0-th link of D_α. For each i in the domain of g_α let $f_\alpha(i)$ be a link of $\tilde{\alpha}$ which contains the image under h_α of the i-th link of C_α. Amalgamate the i-th link of D_α with the $f_\alpha(i)$-th link of $\tilde{\alpha}$. Doing this for each arm α of U produces the desired chain of mesh less than ε which covers M.

Theorem 2. *Every almost chainable homogeneous continuum is a pseudo-arc.*

Proof. This follows immediately from Lemmas 1 through 4 and Bing's result [3] that every chainable homogeneous continuum is a pseudo-arc.

Remark. The discussion in the initial part of this paper gives some background on the homogeneous plane continuum question and one view of the current status of this question and the related question of classifying hereditarily indecomposable homogeneous continua. It does not claim to be comprehensive even for the questions it discusses, and leaves almost totally unmentioned the work on homogeneous continua which are not planar and not hereditarily indecomposable. Since the initial version of this paper was written, Hagopian has shown that no homogeneous tree-like continuum contains an arc, Rogers that every homogeneous hereditarily indecomposable continuum is tree-like, and Lewis that the existence of another homogeneous hereditarily indecomposable continuum implies the existence of one with no nondegenerate chainable subcontinua.

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