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THE ALMOST LINDELÖF PROPERTY FOR BAIRE SPACES**R. A. McCoy and J. C. Smith**

In 1979, B. Scott [6] answered in the affirmative the question (posed by C. E. Aull and J. E. Vaughan) as to whether every pseudocompact metacompact space is compact. This was also shown independently by S. Watson [7]. In his paper Scott also noted that the Mrowka-Isbell space [3, Example 5I] demonstrated that the metacompactness condition could not be weakened to that of θ -refinability. Recently Burke and Davis [1] have shown that the Scott-Watson theorem is true when the metacompactness condition is replaced by the property of σ -paralindelöf. In this paper, we prove an analogous result, Theorem 2, in which the compactness type conditions of the theorem are replaced by Lindelöf type conditions.

We start with some definitions. A space X is *feebly compact* (*feebly Lindelöf*, respectively) provided every discrete family of nonempty open subsets of X is finite (countable, respectively). Also X is *almost compact* (*almost Lindelöf*, respectively) if every open cover of X has a finite (countable, respectively) subfamily whose union is dense in X . On the other hand, an *almost metacompact* space is a space for which every open cover has an open refinement which is point finite on a dense open subset.

Our definition of feebly compact is a slight weakening of the usual one which uses locally finite families instead

of discrete ones; but it is equivalent to the usual definition in a regular space. This property is also called weakly compact or lightly compact. A fact which is used in the proof of Theorem 1 is that a regular feebly compact space is a Baire space [4]. However, it is clear that a regular feebly Lindelöf space need not be a Baire space since every space with the countable chain condition is feebly Lindelöf. So any separable non-Baire space, such as the space of rationals, would be feebly Lindelöf. As for almost compactness, it is easy to see that a regular almost compact space is compact. However, a regular almost Lindelöf space need not be Lindelöf, or even almost metacompact, as shown by the Niemytzki plane [5].

The Scott-Watson theorem can be restated in terms of these more general concepts as follows. The proof is basically the same.

Theorem 1. A regular space is almost compact (and hence compact) if and only if it is feebly compact and almost metacompact.

A natural question is whether Theorem 1 remains true when the compactness properties are replaced by the corresponding Lindelöf properties.

To help answer this question, we make the following observations. First, every feebly Lindelöf, almost metacompact, regular Baire space is almost Lindelöf. On the other hand, as mentioned above, a regular almost Lindelöf space need not be almost metacompact. Therefore we need a

property which is strictly weaker than almost metacompactness in order to obtain our analog. One natural property which might be considered is the "almost metalindelöf" property (every open cover has an open refinement which is point countable on a dense open subset). However, Watson has recently given an example of a pseudocompact metalindelöf space which is not almost Lindelöf¹. Therefore the almost metalindelöf property is still not the "right" property.

We will show that the property that works in obtaining our analog is the following property. A space X is *almost θ -refinable* provided that for every open cover \mathcal{U} of X there exist a sequence $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of open covers of X which refine \mathcal{U} and a dense open subset W of X such that for each $x \in W$, $\text{ord}(x, \mathcal{U}_n) < \infty$ for some n . Note that if $W = X$ for each \mathcal{U} , then X is θ -refinable.

We first establish a preliminary result about feebly Lindelöf spaces².

Lemma. *If X is feebly Lindelöf and \mathcal{U} is an open cover of X which is locally finite on a dense subset of X , then \mathcal{U} contains a countable subfamily whose union is dense in X .*

Proof. Let \mathcal{U} be an open cover of X which is locally finite on dense set D . By Zorn's lemma, the set (which is

¹The authors would like to thank Brian Scott for informing them of Watson's example, and for his proof that this example is not almost Lindelöf.

²The use of this lemma was suggested by the referee in a previous version of this paper.

partially ordered by inclusion) of all families \mathcal{V} of open subsets of X satisfying

- 1) $\text{ord}(\mathcal{V}, \mathcal{U}) < \infty$ for each $\mathcal{V} \in \mathcal{V}$,
- 2) $\text{ord}(\mathcal{U}, \mathcal{V}) \leq 1$ for each $\mathcal{U} \in \mathcal{U}$

has a maximal element \mathcal{M} . It is easy to see that \mathcal{M} is a discrete family since otherwise 2) would be violated. Also \mathcal{M} is countable because X is feebly Lindelöf.

Now define the subfamily \mathcal{U}^* of \mathcal{U} by $\mathcal{U}^* = \{U \in \mathcal{U} \mid U \cap V \neq \emptyset \text{ for some } V \in \mathcal{M}\}$, which is countable since \mathcal{M} is countable and because of 1). To see that $\bigcup \mathcal{U}^*$ is dense in X , suppose not. Then there would exist an $x \in D \cap \overline{(X \setminus \bigcup \mathcal{U}^*)}$. Let W be an open neighborhood of x which is contained in $X \setminus \overline{\bigcup \mathcal{U}^*}$ and which intersects only finitely many members of \mathcal{U} . But $\mathcal{M} \cup \{W\}$ would satisfy both 1) and 2), which contradicts the maximality of \mathcal{M} .

Theorem 2. A regular Baire space is almost Lindelöf if and only if it is feebly Lindelöf and almost θ -refinable.

Proof. The Baire property is only used for the sufficiency. We start with the necessity and assume only that X is regular.

First suppose that X is not feebly Lindelöf. Then there exists an uncountable discrete family $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of nonempty open subsets of X . For each $\alpha \in A$, let V_α be a nonempty open set such that $\overline{V_\alpha} \subseteq U_\alpha$. Let $\mathcal{V} = X \setminus \{\overline{V_\alpha} \mid \alpha \in A\}$, which is open since $\{V_\alpha \mid \alpha \in A\}$ is discrete. Then $\mathcal{U} \cup \{\mathcal{V}\}$ is an open cover of X for which no countable subfamily is dense in X , and thus X is not almost Lindelöf.

Next suppose that X is almost Lindelöf. Let \mathcal{U} be an open cover of X , and let \mathcal{V} be an open cover of X so that $\{\bar{V} \mid V \in \mathcal{V}\}$ refines \mathcal{U} . Since X is almost Lindelöf, \mathcal{V} has a countable subfamily $\{V_n\}_{n=1}^\infty$ whose union W is dense in X . For each n , let $U_n \in \mathcal{U}$ so that $\bar{V}_n \subseteq U_n$, and define $\mathcal{U}_n = \{U_n\} \cup \{U \setminus \bar{V}_n \mid U \in \mathcal{U}\}$. We see that $\{\mathcal{U}_n\}_{n=1}^\infty$ is a sequence of open covers of X refining \mathcal{U} , and that for each $x \in W$ there exists an n such that $x \in V_n$. But then x belongs to only one member of \mathcal{U}_n , so that X is almost θ -refinable.

Finally we show the sufficiency. Let X be a regular Baire space which is both feebly Lindelöf and almost θ -refinable. Let \mathcal{U} be an open cover of X , and let $\{\mathcal{U}_n\}_{n=1}^\infty$ and W be as in the definition of almost θ -refinability. For each n and k , define $A(n,k) = \{x \mid \text{ord}(x, \mathcal{U}_n) \leq k\}$, which is a closed subset of X . Since $W \subseteq \bigcup \{A(n,k) \mid n, k \in \mathbb{N}\}$ and since X is a Baire space, then $\bigcup \{\text{int}[A(n,k)] \mid n, k \in \mathbb{N}\}$ is dense in X . Then for each n and k define $B(n,k) = \text{cl}(W \cap \text{int}[A(n,k)])$; so that $\bigcup \{B(n,k) \mid n, k \in \mathbb{N}\}$ is also dense in X .

Let us fix n and k for the time being and work in the space $B(n,k)$. First, since $B(n,k)$ is the closure of an open subspace of X , it must be feebly Lindelöf. Let Y be the dense subspace $W \cap \text{int}[A(n,k)]$ of $B(n,k)$, which is open in X and is hence a Baire space. Since \mathcal{U}_n is point finite on Y , then it must be locally finite on a dense subset of Y [2]. By the Lemma, \mathcal{U}_n contains a countable subfamily \mathcal{U}^* which covers a dense subset of $B(n,k)$.

Now letting n and k vary, we see that $\bigcup \{\mathcal{U}_n^* \mid n \in \mathbb{N}\}$ is a countable refinement of \mathcal{U} whose union is dense in X . This shows that X is almost Lindelöf.

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