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An example of an indecomposable Hausdorff continuum with exactly one composant and an example of an indecomposable Hausdorff continuum with exactly two composants have been given by David Bellamy [Be]. Such continua cannot be metric continua [M]. We present an example of a hereditarily indecomposable Hausdorff continuum with exactly two composants. Bellamy constructs his one composant indecomposable continuum by identifying two points in different composants of his two composant indecomposable continuum. This technique cannot be used to construct a one composant hereditarily indecomposable continuum. Thus, the problem of the existence of a one composant hereditarily indecomposable continuum remains open.

Definitions and Notations. A continuum is defined to be a compact connected Hausdorff space. Suppose λ is an ordinal, X_a is a topological space for each $a < \lambda$, and if $a < b$ then h_a^b is a mapping from X_b onto X_a so that if $a < b < c < \lambda$ then $h_a^b \circ h_b^c = h_a^c$. Then the space $X = \varprojlim_{a<b<\lambda}\{X_a, h_a^b\}$ denotes the space which is the inverse limit of the inverse system $\{X_a, h_a^b\}_{a<b<\lambda}$. If $\lambda = \omega_0$ and $h_a^{a+1}: X_{\alpha+1} \to X_\alpha$ is defined then h_a^b is defined to be $h_a^{a+1} \circ h_{a+1}^{a+2} \circ \cdots \circ h_{b-1}^b$. Each point P of X is a function from λ into $\cup_{a<\lambda} X_a$ such that $P_a \in X_a$ and $P_a = f_a^b(P_b)$ for all a and b with $a < b$. If

$a < \lambda$ π_a^X denotes the function from X into X_a such that $\pi_a^X(P) = P_a$, the superscript X will be suppressed in some cases when it is clear which space is meant.

The *composant of the continuum* M *containing the point* P *of* M is the set of points Q of M such that there is a proper subcontinuum of M containing P and Q, it is denoted by Cmps(M,P). The continuum M is said to be indecomposable if it is not the union of two proper subcontinua. If X is a space and K is a subset of X then Int(K) denotes the interior of K and Bd(K) denotes the boundary of K.

If X is a metric space, $x \in X$ and $y \in X$, then d will be used to indicate a metric and d(x,y) will be used to indicate the distance from x to y. If $x \in X$ and $\varepsilon > 0$ then $S_\varepsilon(x) = \{t \mid d(x,t) < \varepsilon\}$. If $H \subset X$ and $x \in X$ then $d(x,H) = \text{glb}\{d(x,y) \mid y \in H\}$ and $S_\varepsilon(H) = \{t \mid d(t,H) < \varepsilon\}$.

A *chain* C is a finite sequence of open sets $c_1, c_2, \cdots, c_n$ called links so that $c_i \cap c_j \neq \emptyset$ if and only if $|i-j| \leq 1$. If $\varepsilon > 0$ then the chain $C = c_1, c_2, \cdots, c_n$ is an ε-*chain* means that for each $i = 1,2,\cdots,n$ the diameter of c_i is less than ε. If D is a chain we will also use the notation $D(1), D(2), \cdots, D(n)$ to denote the elements of D. If the chain D covers the continuum M and P and Q are points of M which lie in the first and last links of D respectively then D is said to *cover* M *from the point* P *to the point* Q. Let $\text{mesh}(D) = \text{lub}\{\text{diam } D(i) \mid i = 1,2,\cdots,n\}$ and $D^* = \cup_{i=1}^n D(i)$. If $D = d_1, d_2, \cdots, d_n$ is a chain and $H \subset D^*$ then let D(H) denote the set $\{d_i \mid d_i \cap H \neq \emptyset\}$. Note that if H is a continuum then D(H) is a chain. Suppose that N is a function from the set of positive integers $\{1,\cdots,r\}$ onto the set of positive integers $\{1,\cdots,s\}$. Then N *is a pattern* means that for each integer i with $1 \leq i < r$ we have $|N(i+1) - N(i)| \leq 1$. If N is a pattern $N: \{1,\cdots,r\} \to \{1,\cdots,s\}$, then *the chain* D *follows the pattern* N *in the chain* C means that D has r links, C has s links, and $D(i) \subset C(N(i))$. The chain E is said to be a *consolidation* of the chain D if and only if each link of E is the union of a subcollection of D and each link of D is a subset of some link of E. The chain $D = d_1,d_2,\cdots,d_n$ is *crooked* in the chain $C = c_1,c_2,\cdots,c_m$ means, that if d_i and d_j are links of D lying in the links c_r and c_s respectively of C with $2 < s - r$ then there exist links d_u and d_v so that $d_u \subset c_{s-1}$, $d_v \subset c_{r+1}$ and either $i < u < v < j$ or $i > u > v > j$.

The following theorem is due to Bing [Bi].

Theorem A (Bing [Bi]). *Suppose that* $N: \{1,\cdots,n] \to \{1,\cdots,r\}$ *is a pattern,* N *is onto,* $N(1) = 1$, $N(n) = r$, $D_1,D_2,\cdots$ *is a sequence of chains from the point* P *to the point* Q *such that* D_1 *has* r *links, and for each positive integer* i D_{i+1} *is crooked in* D_i*, and mesh* $(D_i) \leq \frac{1}{i}$. *Then there is an integer* j *and a chain* E *from* P *to* Q *such that* E *is a consolidation of the chain* D_j *and* E *follows pattern* N *in* D_1.

M is a pseudo-arc means that M is a non-degenerate chainable hereditarily indecomposable continuum. Bing has also shown [Bi] that if M is a pseudo-arc then there exists a sequence of chains $D^1, D^2, \cdots$ each covering M such that for $i > 1$ D^{i+1} is crooked in D^i, mesh $D^i < \frac{1}{i}$, and $M = \cap_{i=1}^{\infty} D^{i*}$.

We shall also use the following characterization for hereditarily indecomposable continua which was proven in the metric setting. The theorem is also true in the non-metric case; but we will only need to use it in the metric case.

Theorem B (Krasinkiewicz [K]). *A continuum* X *is hereditarily indecomposable if and only if for each pair* E *and* F *of mutually exclusive closed subsets of* X *and for each open set* U *intersecting all the components of* E *there exist two closed sets* M *and* N *such that*

$X = M \cup N$,

$E \subset M$,

$F \subset N$, *and*

$M \cap N \subset U - (E \cup F)$.

Finally we use an observation made by Cook.

Theorem C (Cook [C]). *If* X *and* Y *are continua,* Y *is hereditarily indecomposable and* h: X → Y *is a continuous map of* X *onto* Y *then* h *is confluent.*

Definition. If h: X → Y is a mapping from X to Y then h is said to be *confluent* if and only if for every subcontinuum I of Y each component of $f^{-1}(I)$ is mapped onto I.

Suppose Y is a pseudo-arc and X is a proper nondegenerate subcontinuum of Y. We wish to construct a retraction of Y onto X with the additional property that Y - X is mapped into a single composant of X. Lemmas 1.1 and 1.2 are technical lemmas necessary for the construction.

Lemma 1.1. *If* X *is a pseudo-arc,* $Q \in X$*, and* Y *is a pseudo-arc properly containing* X *then there exist two closed sets* H *and* K *whose union is* Y *so that* X *is a component of* H, $X \cap K = \{Q\}$, *and* $Bd(H) = Bd(K) = H \cap K$.

Proof. Suppose Y is a pseudo-arc distinct from X containing X. Let $I_1, I_2, \cdots$ be a sequence of pseudo-arcs lying in Y so that $I_{n+1} \subset I_n$ and $X = \cap_{n=1}^{\infty} I_n$. Let P_1 be a point of I_1-X which lies in $S_1(Q)$, and let R_1 be an open set containing P_1 so that $\overline{R}_1 \cap X = \emptyset$ and $\overline{R}_1 \subset S_1(Q)$. Suppose that R_k and P_k have been defined for all $k \leq \ell$. Then let $P_{\ell+1}$ be a point of $I_{\ell+1}$-X which lies in $S_{1/\ell+1}(P) - \cup_{i=1}^{\ell} \overline{R}_i$, and let $R_{\ell+1}$ be an open set containing $P_{\ell+1}$ so that $\overline{R}_{\ell+1} \cap X = \emptyset$, $\overline{R}_{\ell+1} \cap \overline{R}_i = \emptyset$ for $i \leq \ell$, and $\overline{R_{\ell+1}} \subset S_{1/\ell+1}(Q)$. Let $K = \cup_{i=1}^{\infty} \overline{R}_i \cup \{Q\}$, and let $H = \overline{Y - K}$. Then K is closed because $\{\overline{R}_i\}_{i=1}^{\infty}$ is a null sequence with sequential limiting set {Q}. Furthermore $X \cap K = \{Q\}$. The only points of H that are in K are limit points of Y - H, so $Bd(H) = H \cap K = Bd(K)$. Clearly $X \subset H$. Suppose now that X is not a component of H, then let I be the component of H containing X. If $I \neq X$ then there is a point $z \in I - X$, so for some I_n, $z \notin I_n$. But $I \cap I_n \neq \emptyset$ so $I_n \subset I$, but R_n contains a point of I_n and $R_n \subset$ Int K; this is a contradiction. So X is a component of H.

Lemma 1.2. *Suppose* X *is a pseudo-arc,* $Q \in X$, *and* Y *is a pseudo-arc which contains* X *and is the union of two closed point sets* H *and* K *so that* $X \subset H$, $X \cap K = \{Q\}$, *and* $Bd(H) = Bd(K) = H \cap K$. *Suppose further that* D^1 *and* D^2 *are chains covering* X *so that* D^2 *refines* D^1 *and follows the pattern* N *in* D^1, D^1 *covers* Y, *and for* $i = 1,2$ $D^i = d_1^i, d_2^i, \cdots, d_{k_i}^i$, $Q \in d_1^i - \overline{d_2^i}$, $K \subset d_1^i - \overline{d_2^i}$ *and* V *is an open set such that* $K \subset V \subset \overline{V} \subset d_1^i - \overline{d_2^i}$. *Then there is a chain* $A = a_1, a_2, \cdots, a_{k_2}$ *covering* Y *so that* $a_i \cap X = d_i^2 \cap X$, A *follows* N *in* D^1, $K \subset a_1 - \overline{a_2}$ *and* $V \subset a_1$.

Proof. Let U be an open set so that $\overline{V} \subset U \subset \overline{U} \subset d_1^i - \overline{d_2^i}$, $i = 1,2$. By theorem B, Y is the union of two closed sets H^1 and K^1 so that

$$K \cup X \subset K^1$$
$$Y - D^{2*} \subset H^1$$
$$H^1 \cap K^1 \subset V - X \cup K.$$

For each positive integer $n < k_2$, let X_n be a subcontinuum of X irreducible from Q to $\overline{d_{n+1}^2}$ and let N_n be the pattern that $D^2(X_n)$ follows in $D^1(X_n)$ such that $N_n(t) = N(t)$ for all t for which $N_n(t)$ is defined. Note that since $Q \in X_n$ and d_1^1 is the only link of D^1 containing Q then $N_n(1) = 1$. Also note that one of $D^2(X_n)$ and $D^2(X_m)$ is a subchain of the other and they both have the same first link.

Let $G = \{I | I$ is a component of $Y - V - K^1\}$. Each element of G intersects $Bd(V)$, and by definition $G^* \subset H^1$. Since each element of G intersects $Bd(V)$ each element of G must also intersect $d_1^1 - \overline{d_2^1}$.

Let O and R be open sets such that $\mathrm{Bd}(V) \cap H^1 \subset R \subset \overline{R} \subset O \subset \overline{O} \subset U - K^1$. Since $O \subset U$ we have $O \cap d_2^1 = \emptyset$. We can consider that chain $D = O, d_1^1 - \overline{R}, d_2^1, \cdots, d_{k_1}^1 = d_1, d_2, \cdots, d_{k_1+1}$ which covers Y. Let M_n be the function defined as follows:

$$M_n(1) = 1$$

$$M_n(\ell) = N_n(\ell) + 1 \text{ for } 2 \le \ell \le k_2.$$

It can be easily verified that M_n is a pattern, $M_n: \{1,2,\cdots,k_2\} \to \{1,2,\cdots,k_1+1\}$. Note that $M_n(t) = 1$ if and only if $t = 1$, further $M_n(2) = 2$.

For each $I \in G$ there is an integer j_I such that $D^1(I) = D^1(X_{j_I})$. By theorem A there is a chain B_I covering I so that B_I follows M_{j_I} in D. Then, from the definition of D, we have that $B_I(1) \subset O$ and $B_I(\ell) \cap R = \emptyset$ for $\ell > 1$ since R intersects only the first link of D and $M_{j_I}(\ell) = 1$ if and only if $\ell = 1$. Therefore $I \cap \mathrm{Bd}(V) \subset B_I(1)$. For each $I \in G$, Y is the union of two closed sets H_I and K_I so that

$$I \subset H_I,$$

$$(\overline{V} - R) \cup K^1 \cup (Y - B_I^*) \subset K_I,$$

$$H_I \cap K_I \subset (B_I(1) - \overline{B_I(2)}) \cap R - I.$$

Thus $\{H_I | I \in G\}$ covers $Y - V - K^1 - R$. Since $Y - V - K^1 - R$ is compact and $H_I \cap (Y - V - K^1 - R)$ is clopen in $Y - V - K^1 - R$ some finite subcollection $\{H_{I_j}\}_{j=1}^m$ of $\{H_I | I \in G\}$ covers $Y - V - K^1 - R$.

Consider then:

$$J_1 = H_{I_1}$$

$$J_2 = H_{I_2} \cap K_{I_1},$$

$$\vdots$$

$$J_m = H_{I_m} \cap K_{I_{m-1}} \cap K_{I_{m-2}} \cap \cdots \cap K_{I_1}.$$

The collection $\mathcal{J} = \{J_i \cap (Y - V - K^1 - R)\}_{i=1}^m$ is a collection of disjoint sets clopen in $(Y - V - K^1 - R)$ that covers $(Y - V - K^1 - R)$. Suppose $1 \leq r < s \leq m$ then

$$\begin{aligned} J_r \cap J_s &= (H_{I_r} \cap K_{I_{r-1}} \cap \cdots \cap K_{I_1}) \\ &\quad \cap (H_{I_s} \cap K_{I_{s-1}} \cap \cdots \cap K_{I_1}) \\ &\subset H_{I_r} \cap K_{I_r} \\ &\subset R. \end{aligned}$$

So the elements of $\mathcal{J}$ are disjoint.

For notational ease let $B_r = B_{I_r}$. Suppose $r \neq s$ and $(J_r \cap B_r(i)) \cap (J_s \cap B_s(j)) \neq \emptyset$. Then $(J_r \cap B_r(i)) \cap (J_s \cap B_s(j)) \subset J_r \cap J_s \subset R$ so $B_r(i) \cap R \neq \emptyset$ and $B_s(j) \cap R \neq \emptyset$ so $i = 1$ and $j = 1$. Therefore $\{\cup_{r=1}^m J_r \cap B_r(i)\}_{i=1}^{k_i+1}$ is a chain which follows the pattern M_{k_i} in D. Let

$$\begin{aligned} a_1 &= (V \cup R) \cup (d_1^2 \cap K^1) \cup \cup_{r=1}^m (J_r \cap B_r(1)) \\ &\quad \cup \cup_{r=1}^m (J_r \cap B_r(2)) \\ a_j &= (d_j^2 \cap K^1) \cup \cup_{r=1}^m (J_r \cap B_r(j+1)) \text{ if } 1 < j \leq k_2. \end{aligned}$$

Since we have $B_r(\ell) \cap R \neq \emptyset$ if and only if $\ell = 1$ it follows that $a_j \cap (V \cup R) = \emptyset$ for $j > 1$. Then $A = a_1, a_2, \cdots, a_{k_2}$

covers Y and follows N in D^1. Furthermore by construction we have $a_i \cap X = d_i^2 \cap X$ and $K \subset a_1 - \overline{a_2}$, and we also have $V \subset a_1$ since $V \cap \overline{d_2^2} = \emptyset$.

Theorem 1. *Suppose* X *is a pseudo-arc,* X *is irreducible from* P *to* Q, *and* Y *is a pseudo-arc which contains* X *and is the union of two closed point sets* H *and* K *so that* X *is a component of* H, $X \cap K = \{Q\}$, *and* $Bd(H) = Bd(K) = H \cap K$. *Then there is a retraction* h *of* Y *onto* X *so that* $h(K) = Q$, $h^{-1}(P) = P$, *and* h *maps* Y - X *onto the composant of* X *that contains* Q.

Proof. We shall construct the retraction h using the standard technique of covering Y and X with special sequences of chains $\{A^\alpha\}_{\alpha=1}^{\infty}$ and $\{D^\alpha\}_{\alpha=1}^{\infty}$ respectively so that A^α and D^α both follow some pattern N^α in $A^{\alpha-1}$ and $D^{\alpha-1}$ respectively, $\alpha > 1$. Then h is defined by matching the links of the chains $\{A^\alpha\}_{\alpha=1}^{\infty}$ covering Y with the links of the chains $\{D^\alpha\}_{\alpha=1}^{\infty}$ covering X.

Let P be a point of X which is in a composant of X distinct from the one containing Q. Let $\{D^\alpha\}_{\alpha=1}^{\infty}$ be a sequence of chains covering X so that:

i) $D^\alpha = d_1^\alpha, d_2^\alpha, \cdots, d_{k_\alpha}^\alpha$, $D^{\alpha+1}$ refines D^α, and $X = \cap_{\alpha=1}^{\infty} D^{\alpha*}$;

ii) $P \in d_{k_\alpha}^\alpha - \overline{d_{k_\alpha-1}^\alpha}$ and $Q \in d_1^\alpha - \overline{d_2^\alpha}$;

iii) D^1 covers Y, $K \subset d_1^1 - \overline{d_2^1}$, and $K \cap \overline{d_i^\alpha} = \emptyset$ for all α and all $i > 1$;

iv) $\lim_{\alpha\to\infty} \text{mesh } D^\alpha = 0$;

v) mesh $D^{\alpha+1} < \frac{1}{4} d(P, d_{k_\alpha-1}^\alpha)$; and

vi) $d_i^\alpha \cap d_j^\alpha \neq \emptyset$ if and only if $\overline{d_i^\alpha} \cap \overline{d_j^\alpha} \neq \emptyset$.

By condition v we have that the shortest subchain of $D^{\alpha+1}$ containing $d^{\alpha+1}_{k_{\alpha+1}}$ and some link which intersects $d^{\alpha}_{k_\alpha -1}$ has at least 5 links. Let $N^{\alpha+1}$ be a pattern that $D^{\alpha+1}$ follows in D^{α}. Note $N^{\alpha+1}(1) = 1$, $N^{\alpha+1}(k_{\alpha+1}) = k_\alpha$, $N^{\alpha+1}(k_{\alpha+1}-1) = k_\alpha$.

Suppose that ℓ is a positive integer and for each positive integer $\alpha \leq \ell$ we have the following:

1) U^α is an open set containing K such that $K \subset U^\alpha \subset \overline{U^\alpha} \subset U^{\alpha-1}$, $U^\alpha \cap \overline{d^\alpha_i} = \emptyset$ for $i > 1$ and $\overline{U^\alpha} \subset S_{1/\alpha}(K)$;

2) $Y = H^\alpha \cup K^\alpha$ with H^α and K^α closed and $X \subset H^\alpha$, $K^{\alpha-1} \cup (Y - D^{\alpha*}) \subset K^\alpha$, $H^\alpha \cap K^\alpha \subset (d^\alpha_1 - \overline{d^\alpha_2}) \cap \mathrm{Int}(H^{\alpha-1}) \cap U^\alpha - X$; and

3) $A^\alpha - a^\alpha_1, a^\alpha_2, \cdots, a^\alpha_{k_\alpha}$ is a chain covering Y so that if $\alpha > 1$ then A^α follows N^α in $A^{\alpha-1}$, $U_\alpha \subset a^\alpha_1 - \overline{a^\alpha_2}$, $a^\alpha_1 \subset d^\alpha_1$, and $a^\alpha_i \cap H^\alpha = d^\alpha_i \cap H^\alpha$.

Then by condition iii above we can find an open set $U^{\ell+1}$ containing K so that $\overline{U^{\ell+1}} \subset U^\ell$, $U^{\ell+1} \cap \overline{d^{\ell+1}_i} = \emptyset$ for $i > 1$ and $\overline{U^{\ell+1}} \subset S_{1/\ell+1}(K)$. By lemma 1.2 we can find a chain A covering Y so that $a_i \cap X = d^{\ell+1}_i \cap X$, A follows $N^{\ell+1}$ in A^ℓ, $K \subset a_1 - \overline{a_2}$, and $U^{\ell+1} \subset a_1$. By theorem B, since $X \subset \mathrm{Int}\, H^\ell$ and $Q \in (d^{\ell+1}_1 - \overline{d^{\ell+1}_2}) \cap U^{\ell+1} \cap (a_1 - \overline{a_2})$, then there exist closed sets $H^{\ell+1}$ and $K^{\ell+1}$ so that $Y = H^{\ell+1} \cup K^{\ell+1}$, $X \subset H^{\ell+1}$, $K^\ell \cup (Y - D^{\ell+1*}) \subset K^{\ell+1}$, and

$H^{\ell+1} \cap K^{\ell+1} \subset (d^{\ell+1}_1 - \overline{d^{\ell+1}_2}) \cap U^{\ell+1} \cap (a_1 - \bar{a}_2) \cap \mathrm{Int}(H^\ell) - X$.

Let $A^{\ell+1}$ be the chain defined as follows:

$$a^{\ell+1}_1 = \left(a_1 - (H^{\ell+1} \cap \overline{(d^{\ell+1}_2 \cup \cdots \cup d^{\ell+1}_{k_{\ell+1}})})\right) \cup (H^{\ell+1} \cap d^{\ell+1}_1)$$

$$a^{\ell+1}_i = (a_i \cap K^{\ell+1}) \cup (d^{\ell+1}_i \cap H^{\ell+1}) \text{ for } 1 < i < k_{\ell+1}-1$$

$$a^{\ell+1}_{k_{\ell+1}-1} = [(a_{k_{\ell+1}} \cup a_{k_{\ell+1}-1}) \cap K^{\ell+1}] \cup (d^{\ell+1}_{k_{\ell+1}-1} \cap H^{\ell+1})$$

$$a^{\ell+1}_{k_{\ell+1}} = d^{\ell+1}_{k_{\ell+1}} \cap H^{\ell+1}.$$

The fact that $A^{\ell+1}$ is a chain follows from the construction of $H^{\ell+1}$ and $K^{\ell+1}$. The fact that $A^{\ell+1}$ follows $N^{\ell+1}$ in A^{ℓ} follows from the fact that $a^{\ell+1}_1 \subset a^{\ell}_1$, $A^{\ell+1}$ and $D^{\ell+1}$ follow $N^{\ell+1}$ in A^{ℓ} and D^{ℓ} respectively, and from condition v which guarantees that $(a^{\ell+1}_{k_{\ell+1}} \cup a^{\ell+1}_{k_{\ell+1}-1}) \subset a^{\ell}_{k_{\ell}}$. Therefore by induction there exist infinite sequences $\{H^{\alpha}\}^{\infty}_{\alpha=1}$, $\{K^{\alpha}\}^{\infty}_{\alpha=1}$, $\{U^{\alpha}\}^{\infty}_{\alpha=1}$ and $\{A^{\alpha}\}^{\infty}_{\alpha=1}$ satisfying conditions 1-3 above. Furthermore we have $K \subset \cap^{\infty}_{i=1} a^{i}_{1}$ (in fact by a slight modification we can obtain $K = \cap^{\infty}_{i=1} a^{i}_{1}$). We also require that $a^{i}_{k} \cap a^{i}_{\ell} \neq \emptyset$ if and only if $\overline{a^{i}_{k}} \cap \overline{a^{i}_{\ell}} \neq \emptyset$.

Define N^{ℓ}_{m} for $m < \ell$ to be the function $N^{\ell}_{m}\colon \{1,2,\cdots,k_{\ell}\} \to \{1,2,\cdots,k_m\}$ defined by $N^{\ell}_{m} = N^{m} \circ N^{m+1} \circ N^{m+2} \circ \cdots \circ N^{\ell}$. It is easy to see that N^{ℓ}_{m} is a pattern that D^{ℓ} follows in D^{m}.

If $x \in Y$ then let n^{x}_{ℓ} be an integer such that $x \in A^{\ell}(n^{x}_{\ell})$. Let $P^{\ell}_{x} \in X \cap D^{\ell}(n^{x}_{\ell})$.

We shall now construct the retraction h. Claims 1.1 and 1.2 allow us to define h. Claims 1.3 and 1.4 show that h is continuous. Cliam 1.5 shows that h is a retraction. Finally, claims 1.6, 1.7, and 1.8 show that h has the special required properties which will be needed for the construction in Theorem 2.

Claim 1.1. *For each* $x \in Y$ *the sequence* $\{P^{\ell}_{x}\}^{\infty}_{\ell=1}$ *is a Cauchy sequence.*

Proof. Suppose $\varepsilon > 0$. Let N be an integer such that mesh $D^N < \frac{\varepsilon}{3}$. Let ℓ and m be integers larger than N with $m < \ell$. Then $x \in A^m(n_m^x)$ and $x \in A^\ell(n_\ell^x)$. So $|N_m^\ell(n_\ell^x) - n_m^x| \leq 1$. Also we have $P_x^\ell \in D^\ell(n_\ell^x)$, $P_x^m \in D^m(n_m^x)$, $P_x^\ell \in D^m(N_m^\ell n_\ell^x)$, so $d(P_x^\ell, P_x^m) < 2$ mesh $D^m < \varepsilon$. Therefore $\{P_x^\ell\}_{\ell=1}^\infty$ is a Cauchy sequence.

We define $P_x = \lim_{\ell\to\infty} P_x^\ell$.

Claim 1.2. *Suppose* $x \in Y$ *and for each* ℓ Q_x^ℓ *is a point of* $D^\ell(n_\ell^x)$. *Then* $\lim_{\ell\to\infty} Q_x^\ell = P_x$.

Proof. By claim 1.1 $\{Q_x^\ell\}_{\ell=1}^\infty$ is also a Cauchy sequence so it has a sequential limit point. But $d(P_x^\ell, Q_x^\ell) <$ mesh D^ℓ so $\lim_{\ell\to\infty} Q_x^\ell = P_x$.

Define $h: Y \to X$ by $h(x) = P_x$ for all $x \in Y$. By claim 1.2 h is well defined.

Claim 1.3. *If* $x \in Y$ *then* $d(h(x), P_x^k) \leq 2$ mesh D^k.

Proof. From the proof of claim 1.1 we have $d(P_x^\ell, P_x^k) < 2$ mesh D^k. So $\lim_{\ell\to\infty} d(P_x^\ell, P_x^k) \leq 2$ mesh D^k so $d(h(x), P_x^k) \leq 2$ mesh D^k.

Claim 1.4. *h is continuous.*

Proof. Suppose $\varepsilon > 0$. Let N be an integer such that if $m > N$ then mesh $D^m < \frac{\varepsilon}{6}$. Let $m > N$ and let $\delta > 0$ be such that if $d(x,y) < \delta$ then $|n_m^x - n_m^y| \leq 1$. Then $d(h(x), P_x^m) < 2$ mesh D^m, $d(h(y), P_y^m) < 2$ mesh D^m, and $d(P_x^m, P_y^m) < 2$ mesh D^m since $|n_m^x - n_m^y| < 1$. So $d(h(x), h(y)) < 6$ mesh $D^m < \varepsilon$.

Claim 1.5. h *is the identity on* X.

Proof. This follows easily from the fact that for each $x \in X$ P_x^{ℓ} can be chosen to be x for all ℓ. So $P_x = x$.

Claim 1.6. $h^{-1}(P) = P$.

Proof. Suppose not. Then there is some point $z \in Y - X$ such that $h(z) = P$. Since $X = \cap_{i=1}^{\infty} H^i$ it follows that there is some integer j so that $z \notin H^j$. Thus $z \notin a^j_{k_j}$, and $P_z^{j+1} \notin d^j_{k_j} - \overline{d^j_{k_j-1}}$. But $d(P_z^{j+1}, h(z)) \leq 2 \text{ mesh } D^{j+1} < d(P, d^j_{k_j-1})$ by condition v above, so $h(z) \neq P$, this is a contradiction. Hence $h^{-1}(P) = P$.

Claim 1.7. $h(K) = Q$

Proof. $K \subset d_1^{\ell}$ for all ℓ, thus for any $x \in K$ the point P_x^{ℓ} can be chosen to be Q.

Claim 1.8. *If* C *is the composant of* X *containing* Q *then* $h(Y - X) = C$.

Proof. Suppose $z \in Y - X$ and $h(z) \notin C$. Then $z \notin K$ since $h(K) = Q$ and $Q \in C$. Let I be the component of H containing z. Then $I \cap X = \emptyset$ since X is a component of H. Thus $I \cap K \neq \emptyset$, and $P \notin f(I)$ so f(I) is a proper subcontinuum of X and $Q \in f(I)$ since $I \cap K \neq \emptyset$. Thus $f(I) \subset C$. Therefore $h(Y - X) \subset C$.

Suppose that $h(Y - X) \neq C$. Then there is a point $z \in C - h(Y - X)$. Let I be a subcontinuum of C containing Q and z. So I is a proper subcontinuum of X. Let L be a component of H distinct from X, so h(L) is a proper subcontinuum of X, $Q \in h(L)$ but $z \notin h(L)$ therefore $h(L) \subset I$.

So since $h(K) = Q$, $h(Y - X) \subset I$. But $Y - X$ is dense in Y so $h(Y - X)$ must be dense in $h(Y) = X$. This is a contradiction. So the claim is established. This proves the theorem.

In the construction which follows we construct spaces as inverse limits of inverse systems. We construct $X = \varprojlim_{\alpha<\beta<\lambda} \{X_\alpha, h_\alpha^\beta\}$ for some λ so that $X_\alpha \subset X_\beta$ for $\alpha < \beta$ and h_α^β is the identity on X_α. Thus if α is an ordinal and $Y = \{y \mid$ for some $x \in X_\alpha$ and $y_\beta = x$ for all β such that $\alpha < \beta\}$ then Y is a homeomorphic copy of X_α which lies in X. We shall identify this continuum Y with X_α using the natural projection mapping.

The continuum which we seek will be constructed as an inverse limit of an inverse system of pseudo-arcs $\{X_\alpha\}_{\alpha<\omega_1}$, indexed by ω_1 the first uncountable ordinal. The bonding maps between the pseudo-arc X_α and its successor $X_{\alpha+1}$ will be a map of the type guaranteed by Theorem 1. The crucial property of the bonding maps $h_\alpha^{\alpha+1}: X_{\alpha+1} \to X_\alpha$ that will be necessary for the construction is the property that $h_\alpha^{\alpha+1}$ is a retraction that maps $X_{\alpha+1} - X_\alpha$ onto a single composant of X_α. This technique is similar to those employed by Bellamy [B] and Smith [S1,S2].

Theorem 2. *There exists a Hausdorff continuum which is hereditarily indecomposable and which has exactly two composants.*

Proof. We shall first construct countable sequences of pseudo-arcs $\{X_\alpha\}_{\alpha=1}^\infty$ and functions $\{h_\alpha^\beta\}$ and then define the inverse limit $X_{\omega_0} = \varprojlim_{\alpha<\beta<\omega_0} \{X_\alpha, h_\alpha^\beta\}$ (claims 2.1-2.4). Then we shall extend this construction to obtain an inverse system of pseudo-arcs $\{X_\alpha\}_{\alpha<\omega_1}$, so that the inverse limit of the inverse system will be the required continuum (claims 2.4-2.8).

Let X_1 be a pseudo-arc which is irreducible from the point P to the point Q_1 and let X_2 be a pseudo-arc containing X_1 which is the union of two closed point sets H_2 and K_2 so that X_1 is a component of H_2, $X_1 \cap K_2 = \{Q\}$ and $Bd(H_2) = Bd(K_2) = H_2 \cap K_2$. Let h_1^2 be the retraction guaranteed by theorem 1 so that $h_1^2(K_2) = Q_1$, $h_1^{2^{-1}}(P) = P$, and $h_1^2(X_2 - X_1) = Cmps(X_1,Q_1)$. Let $\{M_i^1\}_{i=1}^\infty$ be a monotonic sequence of continua each containing Q_1 and whose union is $Cmps(X_1,Q_1)$. Let $Q_2 \in Int(K_2)$ and $Q_2 \notin Cmps(X_2,Q_1)$. Since h_1^2 is confluent and $h_1^2(Q_2) = Q_1$ then for each positive integer i there is a subcontinuum M_i^2 of X_2 containing Q_2 so that $h_1^2(M_i^2) = M_i^1$, and M_i^2 is clearly a proper subcontinuum of X_2. Let $C_i = Cmps(X_i,Q_i)$ $i = 1,2$. We claim that $C_2 = \cup_{i=1}^\infty M_i^2$ and hence $h_1^2(C_2) = C_1$. For suppose not, then let $z \in C_2 - \cup_{i=1}^\infty M_i^2$. Let I be a proper subcontinuum of M containing Q_2 and z. But each M_i^2 intersects I, so by hereditary indecomposability $M_i^2 \subset I$, and hence $\cup_{i=1}^\infty M_i^2 \subset I$. But $h_1^2(\cup_{i=1}^\infty M_i^2) = C_1$, $I \subset X_2 - X_1$, $h_1^2(X_2 - X_1) = C_1$, and so h(I) is a proper subcontinuum of C_1. This is a contradiction, so $C_2 = \cup_{i=1}^\infty M_i^2$.

By induction for each positive integer i > 1 construct X_i, h_i^{i+1}, K_i, H_i, Q_i, $\{M_j^i\}_{j=1}^\infty$, and C_i so that:

1) X_i is a pseudo-arc, $X_i \subset X_{i+1}$, X_i is irreducible from P to Q_i;

2) X_i is the union of two closed sets H_i and K_i so that $Bd(H_i) = Bd(K_i) = H_i \cap K_i$, X_{i-1} is a component of H_i, $\{Q_i\} = X_{i-1} \cap K_i$, $Int(K_i) \neq \emptyset$;

3) $C_i = Cmps(X_i, Q_i)$, $Q_i \in Int(K_{i-1})$ and $Q_i \notin Cmps(X_i, Q_{i-1})$;

4) $h_i^{i+1}: X_{i+1} \to X_i$ is a retraction of X_{i+1} onto X_i so that $h_i^{i+1}|X_i$ is the identity on X_i, $h_i^{i+1}(K_{i+1}) = Q_i$, $h_i^{i+1}(X_{i+1} - X_i) = C_i$, ${h_i^{i+1}}^{-1}(P) = P$; and

5) $\{M_j^i\}_{j=1}^{\infty}$ is a monotonic set of subcontinua of X_i so that $C_i = \cup_{j=1}^{\infty} M_j^i$, $Q_i \in M_j^i$ for all j, and $h_i^{i+1}(M_j^{i+1}) = M_j^i$ for all j.

Define $X_{\omega_0} = \varprojlim_{i<j<\omega_0} \{X_i, h_i^j\}$ and let $\pi_i: X_{\omega_0} \to X_i$ be the natural projection, $\pi_i(x) = x_i$ where $x = x_1, x_2, \cdots$.

Claim 2.1. X_{ω_0} *is a pseudo-arc.*

Proof. X_{ω_0} is chainable and hereditarily indecomposable.

Claim 2.2. *If* $x \in X_{\omega_0}$ *and there is an integer* $j \geq 2$ *such that* $\pi_j(x) \in X_{j-2}$ *then* $\pi_i(x) \in X_{j-2}$ *for all* $i \geq j$.

Proof. We prove the claim by induction, suppose that $k \geq j$ and $\pi_\ell(x) \in X_{j-2}$ for $j \leq \ell \leq k$. Since $X_{j-2} \subset X_{k-1}$, we have $\pi_k(x) = x_k \in X_{j-2} \subset X_{k-1}$ and by condition 3 since $Q_k \notin Cmps(X_k, Q_{k-1})$ and $Q_{k-1} \in X_{k-1}$ then $X_{k-1} \cap C_k = \emptyset$. But $h_k^{k+1}(X_{k+1} - X_k) = C_k$ so $x_{k+1} \notin X_{k+1} - X_k$ so $x_{k+1} \in X_k$, but h_k^{k+1} is the identity on X_k so $x_{k+1} = x_k$ and hence $x_{k+1} \in X_{j-2}$.

Let Q_{ω_0} be the point $Q_1, Q_2, \cdots$ in the inverse limit space X_{ω_0}. Let $M_j^{\omega_0} = \varprojlim_{i<k<\omega_0} \{M_j^i, h_i^k|_{M_j^k}\}$. Denote $\mathrm{Cmps}(X_{\omega_0}, Q_{\omega_0})$ by C_{ω_0}.

Claim 2.3. $C_{\omega_0} = \cup_{j=1}^{\infty} M_j^{\omega_0}$.

Proof. Suppose $C_{\omega_0} \neq \cup_{j=1}^{\infty} M_j^{\omega_0}$. Since $M_j^{\omega_0}$ is a proper subcontinuum of X_{ω_0} it follows that $\cup_{j=1}^{\infty} M_j^{\omega_0} \subset C_{\omega_0}$. Let $z \in C_{\omega_0} - \cup_{j=1}^{\infty} M_j^{\omega_0}$, then there is a proper subcontinuum I of X_{ω_0} containing z and Q_{ω_0}. There exists an α so that $\pi_\alpha(I) \neq X_\alpha$. But $Q_\alpha \in \pi_\alpha(I)$ so $\pi_\alpha(I) \subset C_\alpha$ and $\pi_\alpha(I) \subset M_j^\alpha$, $\pi_\alpha(I) \neq M_j^\alpha$ for some j. Now $\pi_{\alpha+1}(I)$ contains $Q_{\alpha+1}$ so $\pi_{\alpha+1}(I) \cap M_j^{\alpha+1} \neq \emptyset$. But $h_\alpha^{\alpha+1}(M_j^{\alpha+1}) = M_j^\alpha$ and $M_j^\alpha \not\subset \pi_\alpha(I)$ so $M_j^{\alpha+1} \not\subset \pi_{\alpha+1}(I)$ and so by hereditary indecomposability $\pi_{\alpha+1}(I) \subset M_j^{\alpha+1}$. Therefore by induction $\pi_\lambda(I) \subsetneq M_j^\lambda$ for all $\lambda > \alpha$ and hence $I \subset M_j^{\omega_0}$ which is a contradiction. So $C_{\omega_0} = \cup_{j=1}^{\infty} M_j^{\omega_0}$.

Define: $h_r^s = h_r^{r+1} \circ h_{r+2}^{r+2} \circ \cdots \circ h_{s-1}^s$ for $r < s < \omega_0$

$$h_r^{\omega_0} = \pi_r^{X_{\omega_0}}.$$

Claim 2.4. *If* $r < s$ *then* $h_r^s(X_s - X_r) = C_r$.

Proof. By condition 5 we have $C_{r+1} = \cup_{i=1}^{\infty} M_i^{r+1}$ and $h_r^{r+1}(M_i^{r+1}) = M_i^r$. So $h_r^{r+1}(C_{r+1}) = C_r$. Thus by definition of h_r^s we have $h_r^s(C_s) = C_r$. We prove the claim for the case when $r \neq \omega_0$ by induction. Suppose then that for all ℓ such that $r < \ell < k$ we have $h_r^\ell(X_\ell - X_r) = C_r$. This is clearly true whenever $k = r + 1$. Let $x \in X_{k+1} - X_r$. If $x \notin X_k$

then $x \in X_{k+1} - X_k$ so $h_k^{k+1}(x) \in C_k$; but since $h_r^k(C_k) = C_r$, then $h_r^{k+1}(x) \in C_r$. If $x \in X_k$ then $x \in X_k - X_r$ and h_k^{k+1} is the identity on X_k so $h_r^{k+1}(x) = h_r^k(x)$; and so by the induction hypothesis $h_k^{k+1}(x) \in C_r$. Consider now the case where $r = \omega_0$. Let $x \in X_{\omega_0} - X_r$. Then for some $\delta > r$, $x_\delta \notin X_r$. So $x_\delta \in X_\delta - X_r$. Thus $h_r^{\omega_0}(x) = h_r^\delta \circ \pi_\delta^{X_{\omega_0}}(x) = h_r^\delta(x_\delta) \in C_r$ by the previous case.

We shall now define X_α, Q_α, $\{M_i^\alpha\}_{i=1}^\infty$, C_α and h_β^α (for all $\beta < \alpha$) for all ordinals $\alpha < \omega_1$. Also we shall define H_α and K_α for all non-limit ordinals $\alpha < \omega_1$. Suppose $\delta < \omega_1$ is a limit ordinal and X_λ, etc. have been defined for $\lambda < \delta$ and X_λ is a pseudo-arc for all $\lambda < \delta$. Then let

$$X_\delta = \varprojlim_{\lambda<\gamma<\delta} \{X_\lambda, h_\lambda^\gamma\},$$

Since $\pi_\lambda^{X_\delta}$ is the projection of X_δ onto the λ coordinate we define $h_\lambda^\delta = \pi_\lambda^{X_\delta}$. Let $Q_\delta = \{Q_\lambda\}_{\lambda<\delta}$,

$$M_i^\delta = \varprojlim_{\lambda<\gamma<\delta} \{M_i^\lambda, h_\lambda^\gamma\}, \text{ and}$$

$$C_\delta = \mathrm{Cmps}(X_\delta, Q_\delta).$$

The sets H_δ and K_δ need not be defined for limit ordinals. Since $\delta < \omega_1$ then some countable set is cofinal in δ so X_δ is a pseudo-arc. Furthermore by claim 2.3 $C_\delta = \cup_{i=1}^\infty M_i^\delta$.

Suppose μ is not a limit ordinal but $\mu = \delta + n$ for some limit ordinal δ and some positive integer n, and that X_λ, etc. have been defined for all $\lambda \leq \delta$ and that we have:

1) X_λ is a pseudo-arc,

2) $Q_\lambda \in M_i^\lambda$ and M_i^λ is a subcontinuum of X_λ for all positive integers i, and $C_\lambda = \cup_{i=1}^\infty M_i^\lambda$,

3) if $\beta < \lambda$: a) $h_\beta^\lambda(M_i^\lambda) = M_i^\beta$,

b) $h_\beta^\lambda(X_\lambda - X_\beta) = C_\beta$,

c) $X_\beta \cap C_\lambda = \emptyset$, and

d) $C_\beta = h_\beta^\lambda(C_\lambda)$.

Then we obtain $X_{\delta+n}$, etc. by using the construction above by replacing X_1 with X_δ, Q_1 with Q_δ, and M_i^1 with M_i^δ for all i. Thus conditions 1, 2, and 3 are satisfied for all $\lambda < \omega_1$.

Define $X_{\omega_1} = \varprojlim_{\lambda<\gamma<\omega_1} \{X_\lambda, h_\lambda^\gamma\}$,

$$M_i^{\omega_1} = \varprojlim_{\lambda<\delta<\omega_1} \{M_i^\lambda, h_\lambda^\gamma\}, \text{ and}$$

$$Q_{\omega_1} = \{Q_\lambda\}_{\lambda<\omega_1}.$$

Let $W = \{x|$ there exists $\gamma < \omega_1$ such that if $\alpha > \gamma$ then $\pi_\alpha(x) \in X_\gamma\}$. Thus $W = \cup_{\gamma<\omega_1} X_\gamma$. Let $C_{\omega_1} = \text{Cmps}(X_{\omega_1}, Q_{\omega_1})$. Note, we identify P with $\{P\}_{\lambda<\omega_1}$. If $\gamma < \omega_1$ then X_γ is a proper subcontinuum of X_{ω_1} and $P \in X_\gamma$, so $W \subset \text{Cmps}(X_{\omega_1}, P)$. Since for each $\gamma < \omega_1$ X_γ is irreducible from P to Q_γ then X_{ω_1} is irreducible from P to Q_{ω_1}. We wish to prove that $C_{\omega_1} = X_{\omega_1} - W$. Let $y \in X_{\omega_1} - W$.

Claim 2.5. *If* $\alpha > \beta$ *then* $y_\alpha \notin X_\beta$.

Proof. If $\alpha > \beta$ there exists an ordinal $\delta > \alpha$ such that $y_\delta \notin X_\beta$ (or else $y \in X_\beta \subset W$). Suppose $y_\alpha \in X_\beta$ then $y_\alpha \notin C_\alpha$ by condition 3c above. But $h_\alpha^\delta(X_\delta - X_\alpha) \subset C_\alpha$ so $y_\delta \notin X_\delta - X_\alpha$ so $y_\delta \in X_\alpha$. But h_α^δ is the identity on X_α so $h_\alpha^\delta(y_\delta) = y_\alpha \in X_\beta$ so $y_\delta = y_\alpha$ and $y_\delta \in X_\beta$ which is a contradiction. Furthermore a similar argument establishes the following claim:

Claim 2.6. *If* $\alpha > \beta$ *then there exists* $\delta > \alpha$ *such that* $y_\delta \notin X_\alpha$ *and so* $y_\delta \in X_\delta - X_\alpha$.

Claim 2.7. $y_\alpha \in C_\alpha$.

Proof. By claim 2.5 there is a $\delta > \alpha$ such that $y_\delta \in X_\delta - X_\alpha$. But $h_\alpha^\delta(X_\delta - X_\alpha) \subset C_\alpha$ so $y_\alpha \in C_\alpha$.

Claim 2.8. $y \in C_{\omega_1}$.

Proof. By claim 2.7 and condition 2 above for each α there exists an integer n_α so that $y_\alpha \in M_{n_\alpha}^\alpha$. So there is an uncountable subcollection J of ω_1 and an integer n so that $n_\alpha = n$ for all $\alpha \in J$. Now J is cofinal in ω_1 and $h_\beta^\alpha(M_n^\alpha) = M_n^\beta$ for all $\beta < \alpha$ so $y \in \varprojlim_{\lambda<\gamma<\omega_1} \{M_n^\lambda, h_\lambda^\gamma\}$ and hence $y \in M_n^{\omega_1}$, which is a proper subcontinuum of X_{ω_1} that contains Q_{ω_1}. So $y \in C_{\omega_1}$.

Thus from claim 2.7 we have established that X_{ω_1} has exactly two composants W and C_{ω_1}.

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