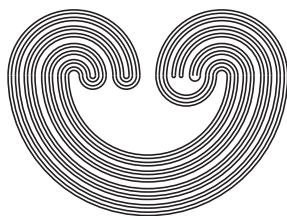

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COUNTABLY PARACOMPACT MOORE SPACES ARE METRIZABLE IN THE COHEN MODEL

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Burke [B] has shown that the Product Measure Extension Axiom implies countably paracompact Moore spaces are metrizable. It should come as no great surprise to those familiar with the result of myself and Weiss [TW] on the normal Moore space conjecture, that adjoining supercompact many Cohen reals also makes countably paracompact Moore spaces metrizable. The proof is not an entirely obvious modification of [TW]. Essential use is made of Burke's topological ideas as well as Dow's work on n -linked "endowments". The same reasons for preferring (or not) the Cohen real proof over the PMEA proof here as for the normal Moore space conjecture--the reader is referred to [TW] and [T] for discussion. In particular, there are generalizations to higher cardinals [T].

By assuming the reader is familiar with [B] and [TW], we are able to make this note pleasantly short, so we shall do so. In particular, to obtain all of Burke's results in the Cohen model, it suffices to prove two lemmas:

Lemma 1. Suppose $\langle X, \mathcal{J} \rangle, \mathcal{Y} \in \mathcal{M}$, $\mathcal{Y}$ a discrete collection in the topological space $\langle X, \mathcal{J} \rangle$. Let G adjoin at least $|\mathcal{Y}|$ many Cohen reals to $\mathcal{M}$. Suppose $\langle X, \mathcal{J}(G) \rangle$ is countably paracompact in $\mathcal{M}[G]$. Then the canonical cover of $\mathcal{Y}$ has in $\mathcal{J}(G)$ an open refinement $\mathcal{H} = \bigcup_{n < \omega} \mathcal{H}_n$ covering $\mathcal{Y}$ such that

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each point in X is locally finite with respect to some $\mathcal{H}_n$. (Recall $\mathcal{J}(G)$ is the topology generated by $\mathcal{J}$ in $\mathcal{M}[G]$). Call such a refinement an eventually locally finite σ -expansion of $\mathcal{Y}$.

Lemma 2. Suppose $\langle X, \mathcal{J} \rangle, \mathcal{Y}, \mathcal{M}$ as in Lemma 1. Let G adjoin any number of Cohen reals to $\mathcal{M}$. If the canonical cover of $\mathcal{Y}$ has an eventually locally finite σ -expansion in $\langle X, \mathcal{J}(G) \rangle$ in $\mathcal{M}[G]$, it does in $\langle X, \mathcal{J} \rangle$ in $\mathcal{M}$.

The essential combinatorial tool in the proof of these lemmas is "n-dowments".

Lemma 3. Let $\mathcal{P}$ be the partial order which adjoins λ many Cohen reals, for arbitrary λ . Then $\mathcal{P}$ is n -dowed, i.e. there is a function F which assigns to each maximal antichain a finite subset of itself, such that if p is any condition with domain of size $\leq n$, and $A_1, \dots, A_n$ are maximal antichains, then there exist $p_i \in F(A_i)$, $i \leq n$, such that $p \wedge \bigwedge_{i \leq n} p_i \in \mathcal{P}$.

Proof. See Dow [D] or modify the proof for $n = 2$ in [TW].

To prove Lemma 1, consider the partial order as $\text{Fn}(\lambda', \omega, \omega)$. Let $G: \lambda' \rightarrow \omega$ be generic. G induces a countable partition of $\mathcal{Y} = \{\mathcal{Y}_\gamma\}_{\gamma < \lambda'}$, $\mathcal{Y}_m = \{\mathcal{Y}_\gamma : G(\gamma) = m\}$. Let

1 $\| \dot{f}: \check{X} \rightarrow \check{J}$ is a neighbourhood assignment which refines a precise locally finite refinement $\{\dot{V}_j\}_{j < \omega}$ of $\{\check{X} - \bigcup_{m \neq j} \dot{\mathcal{Y}}_m : j < \omega\}$.

For each $n \in \omega$ use the n -dowment F_n on maximal antichains $A_x \subseteq \{p: p \text{ decides } \dot{f}(x)\}$ to define neighbourhood assignments $h_n \in \mathcal{M}$, $h_n(x) = \bigcap_{j \leq n} \bigcap \{U: p \Vdash \dot{f}(\check{x}) = \check{U}, p \in F_n(A_x)\}$. Let

$H_{n\gamma} = \cup \{h_n(y) : y \in Y_\gamma\}$. I claim $\{\{H_{n\gamma} : \gamma < \lambda\} : n < \omega\}$ is the required σ -expansion. Suppose on the contrary that no $\{H_{n\gamma} : \gamma < \lambda\}$ is locally finite at x , for some $x \in X$. Let U be a neighbourhood of x , p a condition, and $n \in \omega$ such that $p \restriction \check{U}$ meets $\dot{\check{U}} \dot{\check{V}}_j$'s. Suppose $|\text{dom } p| = k$. U meets $n + 1$ many $H_{(n+k+1)\gamma}$'s say in $h_{n+k+1}(y_{\gamma_j})$, $j \leq n + 1$, $\gamma_j \neq \gamma_j$. Let $\Delta = \{(\gamma_j, j) : j \leq n + 1\}$. There exist $p_j \in F_{n+k+1}(A_{\gamma_j})$, $j \leq n$ such that $q = p \wedge \Delta \wedge p_j \neq 0$. But then $q \restriction \check{U}$ meets $n + 1$ $\dot{\check{U}} \dot{\check{V}}_j$'s, contradiction.

The proof of Lemma 2 proceeds along similar lines-- define a neighbourhood assignment $h_n \in \mathcal{M}$ via n -downment to represent the n 'th level $\dot{f}_n$ of the σ -expansion and then proceed as before to show the h_n 's yield an eventually locally finite σ -expansion.

It is worth noting that Lemma 1 can be used to give another proof of the results of [TW], since we only used the common weakening of countably paracompactness and normality that says a countable discrete collection has a locally finite expansion. This observation nicely compliments the similar one for collectionwise Hausdorff in [W].

[B] should appear before this note; in case [TW] does not, let me briefly indicate why Lemmas 1 and 2 yield

Theorem. Adjoin supercompact many Cohen reals. Then in a countably paracompact space of character $< 2^{\aleph_0}$, discrete collections have eventually locally finite σ -expansions.

The usual product forcing techniques would take care of spaces of cardinality $< 2^{\aleph_0}$, since they would appear at

an initial stage and the remaining Cohen reals would insure that discrete collections had expansions. The supercompact cardinal roughly speaking assures us that if there is forced to be a counterexample, there is forced to be a small one at some initial stage. Lemma 2 preserves the counterexample, but we just saw there were no small counterexamples.

The Theorem by [B] yields

Corollary. Adjoin supercompact many Cohen reals. Then countably paracompact submetacompact regular spaces of character $< 2^{\aleph_0}$ are paracompact.

The easiest way to see this is by using Smith's [S] observation that in a countably paracompact space, discrete collections that have eventually locally finite σ -expansions have locally finite expansions.

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