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by

Yoshio Tanaka and Zhou Hao-xuan

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
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Introduction

Let X be a space and $\mathcal{F}$ be a closed cover of X. Then X *is determined by* $\mathcal{F}$[1], if $A \subset X$ is closed in X whenever $A \cap F$ is relatively closed in F for each $F \in \mathcal{F}$. A space X is a k-*space* (resp. *sequential space*) if it is determined by the cover of all compact subsets (resp. compact metric subsets). Recall that X *is dominated by* $\mathcal{F}$[2], if the union of any subcollection $\mathcal{F}'$ of $\mathcal{F}$ is closed in X and the union is determined by $\mathcal{F}'$. We remark that if X is dominated by $\mathcal{F}$, then it is determined by $\mathcal{F}$, but the converse does not hold. In case that the closed cover $\mathcal{F}$ is increasing and countable, the converse holds. Every CW-complex, more generally every chunk-complex [2] is dominated by a cover of compact metric subsets.

Now, let X be a regular space determined by a point-countable closed cover of separable metric subsets. In [14] and [15], the first author showed respectively that X is Fréchet if and only if it is Lašnev, and that X is metric if and only if it contains no closed copy of a sequential fan S_ω and no Arens' space S_2. Recall that a space X is *Fréchet*, if for every $A \subset X$ and every $x \in \bar{A}$ there

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[1]Following [7], we use "X is determined by $\mathcal{F}$" instead of the usual "X has the weak topology with respect to (or determined by) $\mathcal{F}$.

[2]In some literature, "X has the weak (hereditarily weak; or Whitehead weak) topology with respect to $\mathcal{F}$" is used instead of "X is dominated by $\mathcal{F}$."

exists a sequence in A converging to the point x. Also, a space X is *Lašnev* if X is the closed image of a metric space. The space S_ω is the quotient space obtained from the topological sum of countably many convergent sequences by identifying all the limit points. As for the space S_2, e.g., see [4; Example 1.6.19].

In this paper, we shall give some analogous results ı spaces are dominated by metric subsets, and investigate chunk-complexes and CW-complexes as spaces dominated by compact metric subsets. In Section 1, we show that every Fréchet space dominated by metric subsets is Lašnev, and that every space dominated by metric subsets is metric if and only if it contains no closed copy of S_ω and no S_2. The former is an affirmative answer to a question in [14; Problem 2.4], and the latter was proved in [5] for every space dominated by countably many compact metric subsets. In Section 2, we show that every Fréchet space dominated by compact metric subsets is equivalent to a Fréchet chunk-complex, and such a complex is characterized as the closed image of a locally compact metric space[3]. In Section 3, we show that every Fréchet CW-complex is especially the closed image of the topological sum of Euclidean simplexes (hence, the closed image of a locally compact metric CW-complex).

We assume that all spaces are Hausdorff and all maps are continuous and onto.

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[3]After preparing this paper, M. Itō independently proved that every Fréchet chunk-complex is Lašnev.

1. Spaces Dominated by Metric Subsets and Lašnev Spaces

Lemma 1.1. *Suppose that* X *is a Fréchet space dominated by a closed cover* $\{X_\alpha; \alpha < \beta\}$. *Let* $Y_\alpha = \overline{X_\alpha - \cup_{\delta<\alpha} X_\delta}$, *and* $Y_\alpha^* = Y_\alpha \times \{\alpha\}$. *Let* X^* *be the topological sum* $\sum_\alpha Y_\alpha^*$. *Then the obvious map* $f\colon X^* \to X$ *is closed.*

Proof. Since f is clearly continuous, to prove that f is closed, it is sufficient to show that $\overline{f(F)} \subset f(\overline{F})$ for any $F \subset X^*$. Let $x \in \overline{f(F)} - f(F)$. Since X is Fréchet, there exist distinct points $x_i \in f(F)$ with $x_i \to x$. Choose a point $p_i \in f^{-1}(x_i) \cap F$ for each $i \in N$, and let $P = \{p_i; i \in N\}$. Suppose that P is not contained in any finite union of Y_α^*'s. Let $\alpha_1 = \min\{\alpha; Y_\alpha^* \cap P \neq \emptyset\}$ and choose a point $p_{i_1} \in Y_{\alpha_1}^*$. By induction, we can choose countably many ordinals $\alpha_n = \min\{\alpha; Y_\alpha^* \cap \{p_i; i > i_{n-1}\} \neq \emptyset, \alpha \neq \alpha_1,\cdots,\alpha_{n-1}\}$ and countably many points $p_{i_n} \in Y_{\alpha_n}^*$ with $i_n < i_{n+1}$. Then for each $n \in N$, $\alpha_n < \alpha_{n+1}$ and the subset Y_{α_n} of X contains $x_{i_n} = f(p_{i_n})$. Since X is Fréchet, for each $n \in N$ there exist $y_{nk} \in X_{\alpha_n} - \cup_{\delta<\alpha_n} X_\delta$ with $y_{nk} \to x_{i_n}$. Let $A = \{y_{nk}; n,k \in N\} - \{x\}$. Since $x \in \overline{A}$, there exists a sequence B in A converging to the point x. Let $C = \cup\{X_{\alpha_n}; n \in N\}$. Then $B \subset C$ and $B \cap X_{\alpha_n}$ is finite for each $n \in N$. But the closed subset C of X is determined by $\{X_{\alpha_n}; n \in N\}$. Then B is closed in C, hence in X. Thus, $x \in B$, a contradiction. Hence the subset P of X^* is contained in a finite union of Y_α^*'s. So, infinitely many points p_{n_j} are in some $Y_{\alpha_0}^*$. But $f|Y_{\alpha_0}^*$ is a homeomorphism and $x_{n_j} = f(p_{n_j}) \in Y_{\alpha_0}$.

Thus, since $x_{n_j} \to x$, $p_{n_j} \to y$ for some $y \in Y^*_{\alpha_0}$ with $f(y) = x$. Then, since $p_{n_j} \in f^{-1}(x_{n_j}) \cap F$, $y \in \overline{F}$, so that $x \in f(\overline{F})$. Thus, $\overline{f(F)} \subset f(\overline{F})$. That completes the proof.

By Lemma 1.1, we have the following affirmative answer to the question whether every Fréchet space dominated by compact metric subsets (e.g., Fréchet CW-complex) is Lašnev; see [14; Problem 2.4].

Theorem 1.2. *Every Fréchet space dominated by a closed cover of metric (resp. locally compact metric) subsets is a Lašnev space (resp. the closed image of a locally compact metric space).*

Theorem 1.3. *Let* X *be dominated by a closed cover of Lašnev subsets. Then* X *is Lašnev if and only if it is Fréchet.*

Proof. The "only if" part follows from the easy fact that every Lašnev space is Fréchet. The subsets X_α of X in Lemma 1.1 are Lašnev, so are the subsets Y^*_α of X^*. Thus the "if" part is routinely verified by Lemma 1.1.

Not every countable CW-complex is decomposed into a metric subset and a σ-discrete subset; see [14; p. 284]. But, among Fréchet spaces, we have the following decomposition theorem by Theorem 1.2 and [8; Theorem 2] (resp. [11; Theorem 4]).

Theorem 1.4.* *Every Fréchet space dominated by a closed cover of metric (resp. locally compact metric) subsets is*

*See note added in proof.

decomposed into a metric (resp. locally compact metric) subset and a σ*-discrete (resp. closed discrete) subset.*

In [5], the metrizability of a space dominated by countably many compact metric subsets is characterized by whether or not it contains two copies of the spaces S_ω and S_2. As for the metrizability of a space dominated by metric subsets, analogously we have

Theorem 1.5. *Let* X *be dominated by a closed cover of metric (resp. locally compact metric) subsets. Then* X *is metric (resp. locally compact metric) if and only if it contains no closed copy of* S_ω *and no* S_2.

Proof. The "only if" part is obvious, so we will prove the "if" part. Since X is dominated by metric subsets, by [10; Theorem 3] X is a regular space in which every point is G_δ. Now, X is a sequential space which contains no closed copy of S_ω and no S_2. Thus, by [15; Theorem 3.1], X is strongly Fréchet; that is, if whenever $x \in \overline{A_n}$ with $A_{n+1} \subset A_n$, then there exists $x_n \in A_n$ with $x_n \to x$. Since X is Fréchet, by Theorem 1.2, X is the closed image of a metric (resp. locally compact metric) space. Then, since X is strongly Fréchet, X is metric (resp. locally compact metric) by [9; Corollary 9.10].

In concluding this section, let us consider the products of spaces dominated by metric subsets. For $c = 2^\omega$, the quotient space S_c is similarly defined as the space S_ω.

Theorem 1.6. *Let* X *be dominated by a closed cover of metric subsets. Then the following are equivalent.*

(1) X *is a locally compact metric space.*

(2) $X \times S_c$ *is a* k-*space.*

(3) $X \times S_c$ *is dominated by a closed cover of metric subsets.*

Proof. (3) → (2) is easily proved.

(1) → (3). Let $\mathcal{J}$ be a locally finite cover of compact metric subsets of X, and $\mathcal{C}$ be the obvious cover of the infinite convergent sequences in S_c. Since S_c is dominated by $\mathcal{C}$, from [10; Theorem 1], $X \times S_c$ is dominated by a cover $\mathcal{J} \times \mathcal{C}$ of compact metric subsets.

(2) → (1). Suppose that X contains a closed copy P of S_ω or S_2. Note that the space S_2 is the perfect pre-image of S_ω. Thus, $P \times S_c$ is the perfect pre-image of $S_\omega \times S_c$. Since $P \times S_c$ is a k-space, so is $S_\omega \times S_c$. But, the example [3; p. 563] implicates that $S_\omega \times S_c$ is not a k-space. This contradiction implies that X contains no closed copy of S_ω and no S_2. Then X is metric by Theorem 1.5. Thus each point of $X \times S_c$ is G_δ. Then a k-space $X \times S_c$ is sequential by [9; Theorem 7.3]. Thus a metric space X is locally compact by [12; Theorem 1.1].

Theorem 1.7. *Let* X *be dominated by a closed cover of metric subsets. Then* (1) *and* (2) *below hold.*

(1) *Let* X *be a Fréchet space. Then* X^2 *is a* k-*space if and only if* X *is metric, or* X *is dominated by a countable closed cover of locally compact metric subsets.*

(2) X^ω *is a* k-*space if and only if* X *is metric.*

Proof. (1) Since X is Lašnev by Theorem 1.2, (1) follows from [6; Theorem 2.15].

(2) Let X^ω be a k-space. $(S_\omega)^\omega$ is not a k-space by [12; Proposition 4.2]. While, $(S_2)^\omega$ is the perfect pre-image of $(S_\omega)^\omega$. Then X contains no closed copy of S_ω and no S_2. Hence X is metric by Theorem 1.5.

2. Chunk-Complexes and Closed Images of Locally Compact Metric Spaces

Recall that a *chunk-complex* [2] is a space determined by a cover $\mathcal{H}$ of compact metric subsets, called *chunks*, such that for S, T $\in \mathcal{H}$, either $S \cap T = \emptyset$ or $S \cap T \in \mathcal{H}$, and for $S \in \mathcal{H}$, $\{T \in \mathcal{H};\ T \subset S\}$ is finite. A chunk-complex is an M_1-space [2] dominated by its chunks. Every CW-complex and every locally compact metric space is a chunk-complex. Let us call a chunk-complex *locally countable*, if it has a locally countable cover of chunks.

Lemma 2.1. *Every space determined by a locally countable cover $\mathcal{F}$ of compact metric subsets is a locally countable chunk-complex.*

Proof. For any $A \in \mathcal{F}$, $\{F \in \mathcal{F};\ A \cap F \neq \emptyset\}$ is countable. Then, since X is determined by $\mathcal{F}$, by the proof of (a) $\rightarrow$ (c) of [13; Theorem 1], X is the topological sum of spaces $X_\beta (\beta \in B)$ determined by countably many compact metric subsets $X_{\beta n} (n \in N)$. We can assume that $X_{\beta n} \subset X_{\beta n+1}$ for $\beta \in B$ and $n \in N$. Then X is a locally countable chunk-complex with chunks $X_{\beta n}$.

The regular, separable, non-Lindelöf space Y of [7; Example 9.3] shows that every space determined by a point-finite cover of compact metric subsets is not Lašnev, nor a chunk-complex. But, among Fréchet spaces, we have

Theorem 2.2. *Let* X *be a regular Fréchet space. Then the following are equivalent.*

(1) X *is determined by a point-countable cover of compact metric subsets.*

(2) X *is a locally countable chunk-complex.*

(3) X *is the quotient* s-*image of a locally compact metric space.*

(4) X *is the closed* s-*image of a locally compact metric space.*

Proof. The equivalence (1) $\leftrightarrow$ (3) $\leftrightarrow$ (4) follows from [14; Theorem 2.2] and [1; Theorem 4]. By Lemma 1.1, we have (2) $\rightarrow$ (4).

(4) $\rightarrow$ (2). Let f: L $\rightarrow$ X be a closed s-map with L locally compact metric. Since L is determined by a locally finite cover $\mathcal{J}$ of compact metric subsets, X is determined by a locally countable cover f($\mathcal{J}$) of compact metric subsets. Thus by Lemma 2.1, we have (2).

Every chunk-complex is a space dominated by a cover of compact metric subsets. However, the authors do not know whether the converse holds. So, we shall pose the following question.

Question 2.3. Is every space dominated by a cover of compact metric subsets a chunk-complex?

Concerning the above question, if the cover is point-countable (equivalently, locally countable), then the answer is affirmative by Lemma 2.1. Among Fréchet spaces, the answer is also affirmative without the point-countableness, and in addition, every Fréchet chunk-complex is characterized as the closed image of a locally compact metric space. To prove this, we need the following lemma.

Lemma 2.4. *Let* X *be a Fréchet space dominated by a cover* $\{X_\alpha;\ \alpha < \beta\}$ *of compact subsets, and let* $Y_\alpha = \overline{X_\alpha - \cup_{\delta<\alpha} X_\delta}$. *Then for each* α, *there exists a finite subset* I *such that* $\cup\{Y_\delta \cap Y_\alpha;\ \delta \notin I\}$ *is finite.*

Proof. For some α_o, suppose that $\cup\{Y_\alpha \cap Y_{\alpha_o};\ \alpha \notin I\}$ is infinite for any finite subset I. Then, by induction we can choose countable distinct points $y_n \in X$ and countably many ordinals α_n with $\alpha_n < \alpha_{n+1}$ such that $y_n \in Y_{\alpha_n} \cap Y_{\alpha_o}$ for each $n \in N$. Since $y_n \in Y_{\alpha_n}$, there exists a sequence L_n in $X_{\alpha_n} - \cup_{\delta<\alpha_n} X_\delta$ converging to the point y_n. Since the points y_n are contained in a compact subset X_{α_o}, there exists a sequence $\{y_{n_i};\ i \in N\}$ accumulating to a point $y_o \in X_{\alpha_o}$ with $y_{n_i} \neq y_o$. Then, $y_o \in \overline{\cup\{L_{n_i};\ i \in N\}}$. Thus there exists a sequence $P = \{p_j;\ j \in N\}$ converging to the point y_o such that $p_j \in L_{n_{i(j)}}$ and $p_j \neq y_o$. But, $P \cap X_{\alpha_{n_{i(j)}}}$ is finite for each $j \in N$, so that P is closed in X. Hence, $y_o \in P$, a contradiction.

Theorem 2.5. *The following are equivalent.*

(1) X *is a Fréchet space dominated by a cover of compact metric subsets.*

(2) X *is a Fréchet chunk-complex.*

(3) X *is the closed image of a locally compact metric space.*

Proof. (2) → (3) follows from Theorem 1.2.

(3) → (1). The closed image X of a locally compact metric space has a hereditarily closure-preserving cover[4] of compact metric subsets. Then X is dominated by this cover.

(1) → (2). To show that X is a chunk-complex, let $\mathcal{F} = \{Y_\alpha;\ \alpha < \beta\}$ be a collection of the subsets Y_α in Lemma 2.4; here the Y_α are compact metric. Let $\mathcal{H}$ be the cover consisting of all finite intersections of members of $\mathcal{F}$. We will prove that $\mathcal{H}$ is a cover of chunks for X. Since X is Fréchet, in view of Lemma 1.1, $\mathcal{F}$ is a hereditarily closure-preserving cover of X. Thus X is determined by $\mathcal{F}$. Since $\mathcal{F} \subset \mathcal{H}$, X is determined by $\mathcal{H}$. To show that for each $S \in \mathcal{H}$, $\{T \in \mathcal{H};\ T \subset S\}$ is finite, take any $Y_{\alpha_o} \in \mathcal{F}$ with $S \subset Y_{\alpha_o}$. By Lemma 2.4, there exists a finite subset I_o such that $F_o = \cup\{Y_\alpha \cap Y_{\alpha_o};\ \alpha \notin I_o\}$ is finite. For any $T \subset S$, let $T = \cap\{T_{\alpha_i};\ i = 1,2,\cdots,n\}$. Then all $\alpha_i \in I_o$, otherwise some $\alpha_{i_o} \notin I_o$. The latter case implies that $T \subset Y_{\alpha_{i_o}} \cap Y_{\alpha_o} \subset F_o$, so that T is a subset of the finite

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[4]A cover $\mathcal{F} = \{F_\alpha;\ \alpha \in A\}$ of a space is hereditarily closure-preserving if $\overline{\cup\{A_\alpha\colon \alpha \in A\}} = \cup\{\overline{A}_\alpha\colon \alpha \in A\}$ for any $A_\alpha \subset F_\alpha$.

set F_o. Hence, $\{T \in \mathcal{H}; T \subset S\}$ is finite, because there exist only finitely many subsets of the finite set I_o and only finitely many subsets of the finite set F_o.

Let $f: X \to Y$ be a closed map such that X is dominated by a closed cover $\mathcal{F}$. Then it is easy to show that Y is dominated by $f(\mathcal{F})$. Thus we have the following by Theorem 2.5.

Corollary 2.6. *Every Fréchet space which is the closed image of a chunk-complex is a chunk-complex.*

Corollary 2.7. *Every Fréchet space dominated by a closed cover of chunk-complexes is a chunk-complex.*

Proof. Let X be a Fréchet space dominated by a closed cover $\{X_\alpha; \alpha < \beta\}$ of chunk-complexes, and let $Y_\alpha = \overline{X_\alpha - \cup_{\delta<\alpha} X_\delta}$. Then by Lemma 1.1, X is the closed image of the topological sum of Y_α's. But each closed subset Y_α of a chunk-complex X_α is dominated by compact metric subsets. Since Y_α is Fréchet, by Theorem 1.2, Y_α is the closed image of a locally compact metric space, hence so is X. Thus X is a chunk-complex by Theorem 2.5.

3. CW-Compleses

Let X be a complex. A *subcomplex* L of X is the union of a subset of the cells of X, which are the cells of L, such that, if $e \subset L$ then $\bar{e} \subset L$. Recall that X is a CW-*complex* if it is dominated by the cover of all finite subcomplexes, and each cell is contained in a finite subcomplex. Note that every CW-complex is determined by the cover of all closed cells $\bar{e}$, but not always dominated by this cover.

Now, every CW-complex is a chunk-complex. Thus, by Theorem 2.5, every Fréchet CW-complex is the closed image of a locally compact metric space. But this conclusion can be refined by writing "CW-complex" instead of "space." To show this, we need the following lemma. The proof is essentially the same as in the proof of Lemma 2.4.

Lemma 3.1. *Let* X *be a* CW-*complex with cells* {e}. *If* X *is Fréchet, then for each* $e \in X$ *there is a finite collection* E *of cells such that* $\cup\{\bar{\sigma} \cap e;\ \sigma \in X - E\}$ *is finite.*

Theorem 3.2. *Let* X *be a* CW-*complex. Then the following are equivalent.*

(1) X *is a Fréchet space (resp. Fréchet, locally countable* CW-*complex*[5]*).*

(2) X *is the closed image (resp. closed* s-*image) of a locally compact metric* CW-*complex. (The domain is actually the topological sum of Euclidean simplexes.)*

Proof. Since every Lašnev space is Fréchet, (2) → (1) is obvious. For the parenthetic part, since X is a locally separable space, the CW-complex X is locally countable.

To prove (1) → (2), let X be a Fréchet CW-complex with cells {e}. Let X* be the topological sum $\sum \bar{e}^*$ of $\bar{e}$'s. Since each $\bar{e}^*$ is the closed image of a Euclidean simplex σ,

[5] A CW-complex X is locally countable, if for each $x \in X$ there exists a countable subcomplex A of X such that $x \in \operatorname{int} A$; equivalently, the cover of all closed cells of X is locally countable.

X^* is the closed image of $\sum\sigma$. Thus it suffices to prove that the obvious map $f: X^* \to X$ is closed. Let A be a closed subset of X^*. Then we will prove that $f(A) \cap \overline{d}$ is closed in X for each cell $d \in X$, so that $f(A)$ is closed in X. For $d \in X$, let D be a finite subcomplex of X containing d. Let $A_e = f(A \cap \overline{e}^*) \cap \overline{d}$ for each $e \in X$. Then, $f(A) \cap \overline{d} = \cup\{A_e; e \in X\} = (\cup\{A_e; e \in D\}) \cup (\cup\{A_e; e \notin D\})$. Since each A_e is closed in X, so is $\cup\{A_e; e \in D\}$. Thus it is sufficient to prove that $B = \cup\{A_e; e \notin D\}$ is closed in X. By Lemma 3.1 there is a finite collection F of cells such that $C = \cup\{\overline{e} \cap d'; d' \in D, e \in X - F\}$ is finite. Since $B = \cup\{A_e \cap d'; d' \in D, e \notin D\}$, $B = (\cup\{A_e \cap d'; d' \in D, e \in F - D\}) \cup (\cup\{A_e \cap d'; d' \in D, e \in X - (F \cup D)\}$ Since $\cup\{A_e \cap d'; d' \in D\} = A_e$ is closed in X, so is the first union. The second union is contained in a finite subset $C \cap \overline{d} = \cup\{(\overline{e} \cap d') \cap \overline{d}; d' \in D, e \in X - F\}$. Thus the second union is finite, hence is closed in X. Thus B is closed in X. Hence f is a closed map. That completes the proof.

The proof of the previous theorem suggests the "only if" part of the following. The "if" part is obvious.

Corollary 3.3. *Let* X *be a* CW-*complex with cells* $\{e\}$. *Then* X *is Fréchet if and only if* $\{\overline{e}\}$ *is a hereditarily closure-preserving cover of* X.

Note Added in Proof. The authors have recently shown that every Fréchet space dominated by a closed cover of metric subsets is decomposed into a metric subset and a closed discrete subset.

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Tokyo Gakugei University

Koganei-shi, Tokyo, Japan

and

Sichuan University

Chengdu, China (P.R.C.)