TOPOLOGY PROCEEDINGS

Volume 9, 1984

Pages 165–171

http://topology.auburn.edu/tp/

NON-EXISTENCE OF UNIVERSAL SPACES FOR SOME STRATIFIABLE SPACES

by

Koichi Tsuda

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

NON-EXISTENCE OF UNIVERSAL SPACES FOR SOME STRATIFIABLE SPACES

Kôichi Tsuda

1. Introduction

One of the remarkable properties of metric spaces is that there are universal spaces with a given weight (for example, the Baire's 0-dimensional spaces and generalized Hilbert spaces). The purpose of this paper is to show that there are none for some stratifiable spaces.

Theorem 1. *There are no universal spaces for the following subclasses of stratifiable spaces with a given network weight* $\alpha \geqq \omega$;

(1) σ-*discrete stratifiable spaces*,

(2) M_0-*spaces in the sense of* [6],

(3) *stratifiable* μ-*spaces* [11],

(4) *stratifiable spaces with encircling nets* [15],

(5) *(strongly zero-dimensional) stratifiable spaces*,

(6) *(strongly zero-dimensional)* L-*spaces in the sense of* [13],

(7) *(strongly zero-dimensional) free* L-*spaces* [14].

When we use a given weight instead of a given network weight, we have a similar result:

Corollary 1. *For any given infinite cardinal number* α *there exists a cardinal number* $\beta > \alpha$ *such that there exist no universal spaces for the above classes* (1) $\sim$ (7) *with a given weight* β.

When we consider more restricted subclasses, contained in Lasnev spaces or Nagata spaces, we also have a similar result:

Theorem 2. *There are no universal spaces for either countable Lasnev spaces or first-countable, separable, strongly zero-dimensional stratifiable spaces.*

Remark 1. We have universal spaces for several subclasses of *metric* spaces. In particular, we have:

Theorem 3 [12]. *There exists a universal space for σ-discrete metric spaces with a given weight.*

In this paper all spaces are assumed to be regular T_1. $w(X)$ and $nw(X)$ denote the weight of X and the network weight of X, respectively. A cardinal number is an initial ordinal number, and an ordinal number is the set of its predecessors.

2. Proofs of Theorems

We start with the following proposition.

Proposition 1. *For every infinite cardinal number α there exists a family $\mathcal{S}$ of cardinality $2^{2^{\alpha}}$, none of whose members are homeomorphic to each other, and which consists of σ-discrete stratifiable spaces S with $|S| = nw(S) = \alpha$ and $w(S) \leqq 2^{\alpha}$.*

Proof. Let $\{\alpha_i\}$ be countable disjoint copies of α. Then, for each ultrafilter p on α, let S_p be the space $\alpha_0 \cup \{p\}$, where $\alpha_0 = \cup\{\alpha_i\}$, with the following topology τ_p.

(i) $U \in \tau_p$ if and only if $p \notin U$, or $p \in U$ and there exist an $n \in \omega$ and an $F \in p$ such that $U \cap \alpha_i \supset F$ for every $i \geqq n$.

Then, since each α_i is a clopen discrete subspace of S_p, it is readily seen that S_p is a σ-discrete stratifiable space and that $|S_p| = nw(S_p) = \alpha$ and $w(S_p) \leqq 2^\alpha$. Next, we shall show that

(ii) the family of all S_q's which are homeomorphic to a *fixed* S_p is of the cardinality at most 2^α.

Let S_q and S_r be homeomorphic to a fixed S_p. Then, take *any* homeomorphisms $h_q\colon S_p \to S_q$ and $h_r\colon S_p \to S_r$. To show (ii) it suffices to show that

(iii) $h_q|\alpha_0 \neq h_r|\alpha_0$ if $q \neq r$.

On the contrary, assume that there exists a permutation $h\colon \alpha_0 \to \alpha_0$ such that $h = h_q|\alpha_0 = h_r|\alpha_0$. Since $q \neq r$, we can assume without loss of generality that there exists an $F \in q \setminus r$. Put

$$F_0 = \cup_{i=1}^{\infty} F_i, \text{ where each } F_i \text{ is a copy of } F \text{ in } \alpha_i,$$

and put

$$U_0 = \{q\} \cup F_0.$$

Then, U_0 is a neighborhood of q by the definition (i) of τ_q. Hence, $h_r h_q^{-1}(U_0)$ is a neighborhood of r, and by the definition (i) there exist an $n \in \omega$ and a $G \in r$ such that $G \subset \alpha_i \cap h_r h_q^{-1}(U_0)$ for $i \geqq n$. Hence, $F_n \supset G$, since $\alpha_n \cap h_r h_q^{-1}(U_0) = \alpha_n \cap h\, h^{-1}(U_0) = F_n$. Since F_n is a copy of F and r is an ultrafilter, we have $F \in r$. That contradiction shows that (iii) holds. Since (ii) holds and there exist 2^{2^α} many ultrafilters on α [3, Theorem 3.6.11], there exists a family $\mathcal{S} = \{S_p: p \in \Lambda\}$, with $|\Lambda| = 2^{2^{\alpha}}$, none of whose members are homeomorphic to each other. Thus, our family $\mathcal{S}$ is the required one.

Remark 2. For the case $\alpha = \omega$ we can show the above proposition much easier, using subspaces $\omega \cup \{p\}$ of the Stone-Čech compactification $\beta\omega$ (cf. [13 and 16]).

Proofs of Theorem 1 *and Corollary* 1. The proofs for all the cases of Theorem 1 and Corollary 1 are derived simultaneously from the following observation, since every class (i+1) contains the preceding class (i) except i = 5, the class (3) contains the class (7), and the collection $\mathcal{S}$ in Proposition 1 consists of σ-discrete L-spaces (cf. [4, 5, 7, 8, 9, 11, 13, 14, and 15]).

Let us denote by $[A]^{\kappa}$ the family consisting of all subsets of a set A having cardinality κ. If κ and λ are infinite and $\kappa \leqq \lambda$, then $|[\lambda]^{\kappa}| = \lambda^{\kappa}$. It follows that if X is a stratifiable space (more generally, a paracompact σ-space) with $nw(X) \leqq 2^{\kappa}$, where κ is infinite, then $|X| \leqq |[2^{\kappa}]^{\omega}| = |(2^{\kappa})^{\omega}| = 2^{\kappa}$; consequently, $|[X]^{\kappa}| \leqq 2^{\kappa}$, and we see that X cannot contain copies of all the spaces in Proposition 1, with $\alpha = \kappa$.

Proof of Theorem 2. It suffices to show that there exists a family $\mathcal{J}$ (respectively, $\mathcal{U}$) of cardinality 2^{c}, where $c = 2^{\omega}$, none of whose members are homeomorphic to each other, and which consists of countable Lasnev spaces (respectively, σ-compact, first-countable, separable, strongly zero-dimensional stratifiable spaces).

Let C be the Cantor set in the real line with usual Euclidean topology, and let $0 \in C$. Take a countable dense subset D of C, and let $\{x_n\} \subset C$ be a sequence converging to 0. At first, we show the existence of $\mathcal{J}$. Put

$$F = \{0\} \times C, \text{ and } A = \{(x_n, d_k): k \leq n, \text{ and } n \in \omega\},$$

where $D = \{d_k: k \in \omega\}$. Then, A is a countable discrete subset in C^2 with $\mathrm{cl}(A) \supset F$. For every subset $T \subset F$ let

$$Y_T = T \cup A.$$

Take a point $p_T \notin A$, and let $X_T = \{p_T\} \cup A$. Define a function $\phi_T: Y_T \to X_T$ as $\phi_T(T) = p_T$ and $\phi_T(a) = a$ for every $a \in A$. We topologize the set X_T as

$$U \text{ is open in } X_T \text{ if } \phi_T^{-1}(U) \text{ is open in } Y_T.$$

By the definition of ϕ_T, ϕ_T is a closed map between Y_T and X_T. Thus, each X_T is a countable Lasnev space. By the proof of Proposition 1, one can show that there exists a family $\mathcal{J} = \{X_T: T \in \mathcal{F}\}$, with $|\mathcal{F}| = 2^c$, none of whose members are homeomorphic to each other. Next, we show the existence of $\mathcal{U}$. We shall modify the examples constructed in [2]. Let $\mathcal{F}$ be a free ultrafilter on ω. Then, enumerate the family of all elements of $\mathcal{F}$ as $\{F_s: s \in C\}$, and for each $s \in C$ and $m \in \omega$ choose a $q_s(m) \in D$ such that $0 < |s - q_s(m)| < 1/m$, and put

$$D_s = \{q_s(m): m \in \omega\}, \text{ and } E_s = \{x_n: n \in F_s\}.$$

We topologize the set

$$\Delta_{\mathcal{F}} = (C \times \{0\}) \cup (D \times \{x_n: n \in \omega\})$$

as follows. Points of $D \times \{x_n: n \in \omega\}$ are isolated, and basic neighborhoods of a point $(s,0) \in C \times \{0\}$ have the form

$$B_m(s) = \{(x,y) \in \Delta_{\mathcal{F}}: |s-x| < 1/m\} \setminus (D_s \times E_s \cup \{s\} \times \{x_n: n \in \omega\})$$

for $m \in \omega$. Then, it is known [2] that each $\Delta_{\mathcal{J}}$ is first-countable σ-compact, strongly zero-dimensional, cosmic. We can show without difficulty that each $\Delta_{\mathcal{J}}$ is stratifiable. Here, we show moreover that it admits a free L-structure. Let $\mathcal{V} = \{V_i : i \in \omega\}$ be a countable clopen base of C. Put

$$L_0 = C \times \{0\},\ L_{2i} = V_i \times C, \text{ and } L_{2i+1} = \{u_i\},$$

where $\Delta_{\mathcal{J}} \setminus L_0 = \{u_i : i \in \omega\}$. Put

$$\mathcal{U}_{L_0} = \{\{u\} : u \notin L_0\}, \text{ and } \mathcal{U}_{L_i} = \{\Delta_{\mathcal{J}} \setminus L_i\}$$

for each $L_i \neq L_0$. Then, one can easily check that $(\mathcal{L} = \{L_i : i \in \omega\}, \mathcal{U}_L : L \in \mathcal{L})$ is a free L-structure of $\Delta_{\mathcal{J}}$. Again, by the proof of Proposition 1, we can show that there exists a family $\mathcal{U} = \{\Delta_{\mathcal{J}} : \mathcal{J} \in \Lambda\}$, with $|\Lambda| = 2^c$, none of whose members are homeomorphic to each other. That completes the proof of Theorem 2.

Professor Junnila [10] has kindly communicated to the author the following problem as well as the reference [12] of Theorem 3.

Problem. Does there exist a universal space for closed images of *countable metrizable* spaces?

Added in Proof. By the proof of Theorem 2 it follows that there are no universal spaces for separable Lasnev spaces.

References

1. C. R. Borges, *On stratifiable spaces*, Pacific J. Math. 17 (1966), 1-16.
2. E. K. van Douwen and T. C. Przymusiński, *First countable and countable spaces all compactifications of which contain* βN, Fund. Math. 102 (1979), 229-234.

3. R. Engelking, *General Topology*, PWN, Warszawa, 1977.
4. G. Gruenhage, *Stratifiable spaces are* M_2, Top. Proc. 1 (1976), 221-226.
5. ________, *Stratifiable* σ*-discrete spaces are* M_1, Proc. Amer. Math. Soc. 72 (1978), 189-190.
6. R. W. Heath and H. J. K. Junnila, *Stratifiable spaces as subspaces and continuous images of* M_1*-spaces*, Proc. Amer. Math. Soc. 83 (1981), 146-148.
7. M. Ito, M_0*-spaces are* μ*-spaces*, Tsukuba J. Math. 8 (1984), 77-80.
8. H. J. K. Junnila, *On strongly zero-dimensional* F_σ*-metrizable stratifiable spaces*, in: *Topology* (Proceedings, Leningrad 1982), Springer Lecture Notes 1060 (1984), 67-75.
9. ________, *A characterization of* M_0*-spaces*, Proc. Amer. Math. Soc. 91 (1984), 481-484.
10. ________, A letter of August 29, 1984.
11. T. Mizokami, *On* M*-structures*, Top. Appl. 17 (1984), 63-89.
12. A. Mysior, Handwritten manuscript.
13. K. Nagami, *The equality of dimension*, Fund. Math. 106 (1980), 239-246.
14. ________, *Dimension of free* L*-spaces*, Fund. Math. 108 (1980), 211-224.
15. S. Oka, *Dimension of stratifiable spaces*, Trans. Amer. Math. Soc. 275 (1983), 231-243.
16. S. Shelah and M. E. Rudin, *Unordered types of ultrafilters*, Top. Proc. 3 (1978), 199-204.

Ehime University

Matsuyama 790, Japan