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ON PERFECT IRREDUCIBLE PREIMAGES

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0. Introduction

In this paper we want to consider, for a given space X , the set $\xi^*(X)$ of perfect irreducible preimages of X . (A perfect map is assumed to be continuous.) The elements of $\xi^*(X)$ are pairs (Y, f) , where Y is a space and $f: Y \rightarrow X$ is such a map. $\xi^*(X)$ has a natural partial order \leq and in this partial order $\xi^*(X)$ has a largest element, the so-called absolute of X , to be denoted by $(E X, \pi)$.

In the literature the elements of $\xi^*(X)$ appeared under various names, such as *covers* [Ba], *resolutions* [Fr] and in my dissertation [Ve₁] they were called (projective) *expansions*, a term I also will use in this paper.

Some interesting expansions of a space X are known in the literature. In [D,H,H] it was proved that for a compact space X , there always exists a smallest quasi-F expansion (the notion of a quasi-F space will be defined in section 1) and in [Ve₂] I proved there exists a smallest basically disconnected expansion. Both results were obtained using algebraic methods. One aim of this paper is to show that no algebraic methods are needed at all. Furthermore we will generalize this, by defining the notion of an expansion property. The idea behind this is the following. In my opinion expansions behave very much like extensions. In the case of extensions one enlarges the space by adding points, for expansions one "splits" the points of X . The

second aim of this paper is to introduce a notion dual to the notion of an extension property, due to Herrlich and v.d. Slot [H,S]. The expansion property notion I defined is only an attempt. I did not succeed in defining it in such a way that a real link can be constructed between expansions of extensions and vice versa (for a more precise formulation, see 2.10), as for example in the formula $\beta EX = E\beta X$. Finally we mention that space will always mean *Hausdorff* space.

1. Some Remarks on the Lattice of Expansions

In this section we state some results about expansions; we do not go in much detail. Recall that a surjective map $f: Y \rightarrow X$ is called *irreducible*, if $f(A) \neq X$ for every proper closed subset $A \subset Y$. Let X be a space. An *expansion* of X is a pair (Y, f) of a space Y and a perfect irreducible map $f: Y \rightarrow X$. Two expansions (Z, g) and (Y, f) of X are called *equivalent*, if there is a homeomorphism $h: Z \rightarrow Y$ such that $f \circ h = g$. Let $\mathcal{E}^*(X)$ denote the set of (non-equivalent) expansions of X . On $\mathcal{E}^*(X)$ there is a natural partial order \leq , namely: $(Z, g) \leq (Y, f)$ iff there is a continuous map $h: Y \rightarrow Z$ such that $g \circ h = f$. (Observe that h is perfect irreducible.) It is known that $\mathcal{E}^*(X)$ has a largest element. If X is regular, then the largest element is the *Iliadis absolute* $(E X, \pi)$. For arbitrary Hausdorff spaces the largest element is the so-called Banaschewski absolute [Ba]. For an excellent survey on absolutes, see [Wo]. We will restrict our attention to regular spaces. Observe that the underlying space of an expansion of a regular space is

regular, since perfect preimages of regular spaces are regular. We start with the following.

1.1 *Proposition.* $[Ve_1]$ $(\xi^*(X), \leq)$ is a complete lattice.

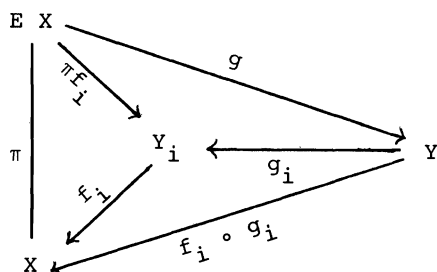
Probably this proposition is well known in the literature but I do not have any references for this proposition, therefore I shall indicate a proof. First we need two lemmas, whose proofs are immediate.

1.2 *Lemma.* Let X be a space and let $\{\mathcal{P}_i\}_i$ be a collection of compact partitions of Y . For $x \in X$, let $P_i(x)$ denote the unique element of \mathcal{P}_i containing x . Put $P(x) = \bigcap_i P_i(x)$. Then $\mathcal{P} = \{P(x) : x \in X\}$ is a compact partition of X and \mathcal{P} is the coarsest partition which refines each \mathcal{P}_i .

1.3 *Lemma.* Let $f: Z \rightarrow X$ be a perfect, irreducible surjection. If Y is a space and $h: Z \rightarrow Y$ and $g: Y \rightarrow X$ are continuous maps such that $f = g \circ h$, then both g and h are perfect irreducible maps.

Proof of Proposition 1.1. Obviously $\xi^*(X)$ has a smallest element, namely (X, id) . Therefore it suffices to show that each subset of $\xi^*(X)$ has a supremum. Take a subset $\{(Y_i, f_i) : i \in I\}$ of $\xi^*(X)$. For each $i \in I$ there is a map $\pi f_i: E X \rightarrow Y_i$ such that $\pi = f_i \circ \pi f_i$. Let \mathcal{P}_i be the compact partition of EX defined by $\mathcal{P}_i = \{\pi f_i^{-1}(y) : y \in Y_i\}$. Consider the compact partition \mathcal{P} of EX , as described in lemma 1.2. Observe that $\forall x, y \in EX$: either $P(x) \cap P(y) = \emptyset$

or $P(x) = P(y)$. In fact, if $P(x) \neq P(y)$, then $\exists i \in I$ such that $P_i(x) \cap P_i(y) = \emptyset$. Observe that P is the partition corresponding to the equivalence relation: $x \sim y \Leftrightarrow \pi f_i(x) = \pi f_i(y) \ (\forall i \in I)$.[⊗] Using [⊗] it is easy to see that the space $EX \text{ mod } \mathcal{P}$ (with quotient topology) is Hausdorff. Furthermore, $EX \text{ mod } \mathcal{P} = Y$ maps onto Y_i by a map $g_i: Y \rightarrow Y_i$ such that $\pi f_i = g_i \circ g$ (here $g: EX \rightarrow Y$ denotes the identification map).



Since $f_i \circ g_i \circ g = \pi (\forall i \in I)$, it follows that the map $f_i \circ g_i: Y \rightarrow X$ does not depend on $i \in I$. Put $k = f_i \circ g_i$, then it is easy to see that $\sup \{(Y_i, f_i): i \in I\} = (Y, k)$.

Let us assume that X is a Tychonoff space. Unfortunately this does not imply that the elements of $\mathcal{E}^*(X)$ have Tychonoff underlying spaces. See for example [H,I] or [H,V].

1.4 Definition. Let X be a regular space. Then $\mathcal{E}(X)$ denote the set of Tychonoff expansions of X .

Observe that $\mathcal{E}(X) \neq \emptyset$, since $(EX, \pi) \in \mathcal{E}(X)$. The relative order on $\mathcal{E}(X)$ need not induce a lattice structure on $\mathcal{E}(X)$. In [Ve₁] I constructed an example of a non-Tychonoff regular space X which has two expansions (Y_i, f_i) ($i = 1, 2$)

in $\mathcal{E}(X)$, such that $\inf((Y_1, f_1), (Y_2, f_2)) = (X, \text{id})$ (infimum is taken in $\mathcal{E}^*(X)$). Obviously in $\mathcal{E}(X)$, (Y_1, f_1) and (Y_2, f_2) have no infimum.

1.5 Lemma. $[\text{Ve}_1]$ Let X be a regular space. If $\{(Y_i, f_i) : i \in I\} \subset \mathcal{E}(X)$, then $\sup\{(Y_i, f_i)\}_i \in \mathcal{E}(X)$ (supremum is taken in $\mathcal{E}^*(X)$).

1.6 Corollary. Let X be a Tychonoff space. Then $(\mathcal{E}(X), \leq)$ is a complete lattice.

Proof. $(\mathcal{E}(X), \leq)$ has a smallest element (X, id) .

Lemma 1.5 shows that every subset of $\mathcal{E}(X)$ has a supremum in $\mathcal{E}(X)$. The conclusion follows.

I do not know whether there exists a space X such that $\mathcal{E}(X)$ is a lattice, but not a complete one.

We recall the following definition.

1.8 Definition. Let X be a Tychonoff space.

(i) X is called *basically disconnected*, if the closure of every cozero set is open.

(ii) X is called *quasi-F*, if each dense cozero set is C^* -embedded.

1.9 Theorem. Let X be a Tychonoff space. Then

(i) $[\text{Ve}_2]$ There exists a smallest element $(\Lambda X, \Lambda)$ among the basically disconnected expansions of X .

(ii) $[D, H, H]$ If X is compact then there exists a smallest element (FX, F) among the quasi-F expansions of X .

1.10 *Definition.* [H,S] An *extension property* E , is a topological property such that compact space have E and such that P is productive and closed hereditary.

1.11 *Theorem.* [H,S] Let X be a Tychonoff space and E an extension property. Then each space X has a largest E -expansion eX . (Recall, an E -extension is an extension with property E .) The extension eX is the unique extension with property E and the additional property: if αX is any E -extension of X then $\text{id}: X \rightarrow \alpha X$ can be extended continuously to a map $e(\text{id}): eX \rightarrow \alpha X$. (In fact, any map $f: X \rightarrow T$, where T has property E , can be extended to a map $ef: eX \rightarrow T$.)

Observe that the previous theorem defines an epi-reflection $X \rightarrow eX$ from the category of Tychonoff spaces and continuous maps to the category of spaces with property E and continuous maps.

2. Expansion Properties

In this section all spaces are assumed to be Tychonoff. (However, for most of the results on the lattice $\mathcal{C}(X)$, such as theorem 2.3, it is easy to formulate corresponding statement for the lattice $\mathcal{C}^*(X)$.) We want to consider properties of open subsets of topological spaces, properties which will depend on the "way" in which the subspace under consideration is embedded in X . For example, if one wants to decide whether an open subset U of X is Lindelöf, then neither the embedding $\text{id}: U \rightarrow X$ nor the space X are relevant. However, to decide whether a particular open $U \subset X$ is a cozeroset in X , then more than "only U " is needed.

2.1 Definition. An *expansion property* \mathcal{P} is a property which particular open subsets of a space can or cannot have in that space, with the following restriction: If $f: X \rightarrow Y$ is a perfect irreducible map and an open subset U has property \mathcal{P} in Y , then $f^{-1}(U)$ has property \mathcal{P} in X .

Formally one can consider an expansion property to be a class \mathcal{P} of pairs $\{(U, X): X \text{ a space and } U \text{ open in } X\}$, satisfying the additional property: If $(U, Y) \in \mathcal{P}$ and $f: X \rightarrow Y$ is a perfect irreducible map, then $(f^{-1}(U), X) \in \mathcal{P}$.

Observe that: if $h: X \rightarrow Y$ is an homeomorphism and U is an open subset of X , then U has \mathcal{P} in X iff $h(U)$ has \mathcal{P} in Y .

2.2 Definition. Let \mathcal{P} be an expansion property. Let X be a space.

(i) X is called *\mathcal{P} -disconnected* if each open $U \subset X$ with \mathcal{P} in X has clopen closure.

(ii) X is called *strongly \mathcal{P} -disconnected* if each open $U \subset X$ with \mathcal{P} in X is C^* -embedded and has clopen closure.

(iii) X is called *quasi- \mathcal{P}* if each dense and open $U \subset X$ with \mathcal{P} in X is C^* -embedded.

The following theorem is the main result of this paper.

2.3 Theorem. Let \mathcal{P} be an expansion property. Let X be a space.

(i) There exists a smallest element (pX, p) among the \mathcal{P} -disconnected expansions of X .

(ii) There exists a smallest element (p^*X, p^*) among the strongly \mathcal{P} -disconnected expansions of X .

(iii) *There exists a smallest element $(\tilde{p}X, \tilde{p})$ among the quasi- \mathcal{P} expansions of X .*

(iv) *The set of \mathcal{P} -disconnected (strongly \mathcal{P} -disconnected) (quasi- \mathcal{P}) expansions is a complete lattice (in its relative order).*

Before we give a proof of the theorem, let us consider some examples.

2.4 *Examples.* (i) Define: (an open) $U \subset X$ has \mathcal{P} iff U is open in X . Obviously: X is extremally disconnected $\Leftrightarrow X$ is \mathcal{P} -disconnected $\Leftrightarrow X$ is strongly \mathcal{P} -disconnected $\Leftrightarrow X$ is quasi- \mathcal{P} .

Therefore, $(EX, \pi) = (pX, p) = (p^*X, p^*) = (\tilde{p}X, \tilde{p})$, for every space X .

(ii) $U \subset X$ has \mathcal{P} iff U is clopen. Then, every space is \mathcal{P} -disconnected, strongly \mathcal{P} -disconnected and quasi- \mathcal{P} .

Therefore, $(X, id) = (pX, p) = (p^*X, p^*) = (\tilde{p}X, \tilde{p})$ for every space X .

(iii) $U \subset X$ has \mathcal{P} iff U is a cozeroset. Then, X is basically disconnected iff X is \mathcal{P} -disconnected. X is quasi- \mathcal{P} iff X is a quasi-F space. Therefore $(\Lambda X, \Lambda) = (pX, p) = (p^*X, p^*)$ and $(\tilde{p}X, \tilde{p}) = (FX, F)$ (see 1.9).

(iv) $U \subset X$ has \mathcal{P} iff U is Čech-complete (and open). Since open subsets of complete spaces are complete and expansions of complete spaces are complete, it follows that $(EX, \pi) = (pX, p) = (p^*X, p^*) = (\tilde{p}X, \tilde{p})$, provided that X is complete. However, a space as the rationals \mathbb{Q} have no (non-empty) complete open subsets, therefore \mathbb{Q} is (strongly) \mathcal{P} -disconnected, and quasi \mathcal{P} . It follows that

$$(Q, id) = (pQ, p) = (p^*Q, p^*) = (\tilde{p}Q, \tilde{p}).$$

(v) $U \subset X$ has \mathcal{P} iff $c\mathcal{L}U$ is not connected. Then \mathcal{P} is an expansion property and every connected space is quasi- \mathcal{P} . If one considers $X = [0, 1]$ then $(X, id) = (\tilde{p}X, \tilde{p})$. It is easy to see that $(pX, p) = (p^*X, p^*) = (EX, \pi)$ (if $X = [0, 1]$).

The list of expansion properties seems endless.

Proof of the Theorem 2.3. (i) The idea is to build an inverse limit $\{(p_\alpha X, p_\alpha) : \alpha \in \text{ord}\}$ of expansions of X , such that: $\alpha < \beta \Rightarrow (p_\alpha X, p_\alpha) \leq (p_\beta X, p_\beta)$. Then $(pX, p) = \lim_{\leftarrow} \{(p_\alpha X, p_\alpha) : \alpha \in \text{ord}\}$. We first construct $(p_1 X, p_1)$.

Consider $\mathcal{U} = \{U \subset X : U \text{ open and } U \text{ has } \mathcal{P} \text{ in } X\}$.

For $U \in \mathcal{U}$, an expansion (X_U, f_U) of X is defined by

$$X_U = c\mathcal{L}U \oplus c\mathcal{L}(X - c\mathcal{L}U) \quad (= \text{disjoint top. sum})$$

$$f_U(x) = x.$$

Then (X_U, f_U) is an expansion of X .

Claim. If (Y, f) is a \mathcal{P} -disconnected expansion of X and if $U \in \mathcal{U}$ then $(Y, f) \geq (X_U, f_U)$.

Indeed, U has \mathcal{P} in X , hence $f^{-1}(U)$ has \mathcal{P} in Y , which is \mathcal{P} -disconnected, i.e. $c\mathcal{L}(f^{-1}(U))$ is clopen in Y . It follows that $f(c\mathcal{L}f^{-1}(U)) = c\mathcal{L}U$ and $f(Y - c\mathcal{L}f^{-1}(U)) = c\mathcal{L}(X - c\mathcal{L}U)$. Since $Y = c\mathcal{L}f^{-1}(U) \oplus (Y - c\mathcal{L}f^{-1}(U))$, it readily follows that there is a map $g: Y \rightarrow X_U$ such that $f_U \circ g = f$, i.e. $(X_U, f_U) \leq (Y, f)$. Define $(p_1 X, p_1) = \sup\{(X_U, f_U) : U \in \mathcal{U}\}$. Observe that X is \mathcal{P} -disconnected iff $(p_1 X, p_1) = (X, id)^\oplus$. Evidently this statement does not imply that $p_1 X$ is \mathcal{P} -disconnected. The claim implies that $(Y, f) \geq (p_1 X, p_1)$, for each \mathcal{P} -disconnected expansion (Y, f) of X . In particular,

each \mathcal{P} -disconnected expansion of X can be considered as a \mathcal{P} -disconnected expansion of $(p_1 X, p_1)$. Consider the above construction as an operator which assigns to a space Z an expansion $(p_1 Z, p_1)$ (i.e. $p_1: p_1 Z \rightarrow Z$).

Define $(p_\alpha X, p_\alpha)$ by induction:

$$\begin{aligned}(p_{\alpha+1} X, p_{\alpha+1}) &= (p_1(p_\alpha X), p_\alpha \circ p_1) \\ (p_\alpha X, p_\alpha) &= \sup_{\beta < \alpha} (p_\beta X, p_\beta) \quad (\alpha \text{ a limit ordinal}).\end{aligned}$$

By induction it follows (see *): if (Y, f) is a \mathcal{P} -disconnected expansion of X , then $(Y, f) \geq (p_\alpha X, p_\alpha)$, for all α . (**)

But $\zeta(X)$ is a set. It follows that the $-$ in $(\zeta(X), \leq)$ - increasing sequence $\{(p_\alpha X, p_\alpha): \alpha \in \text{ord}\}$ must become constant. If $(p_{\alpha+1} X, p_{\alpha+1}) = (p_\alpha X, p_\alpha)$, then (*) implies that $(p_\alpha X, p_\alpha)$ is \mathcal{P} -disconnected and (**) implies that $(p_\alpha X, p_\alpha)$ is the smallest \mathcal{P} -disconnected expansion of X .

(ii) Can be proved in the same way. For all $U \in \mathcal{U}$, consider the map $\text{id}: U \rightarrow X$. This map has an extension $\beta(\text{id}): \beta U \rightarrow \beta X$. Put $U^* = \beta(\text{id})^{-1}(c\mathcal{L}_X U)$. Then $U \subset U^* \subset \beta U$ and the map $\beta(\text{id}): U^* \rightarrow c\mathcal{L}_X U$ is perfect irreducible.

Define the expansion (X_U^*, f_U^*) by:

$$\begin{aligned}X_U^* &= U^* \oplus c\mathcal{L}(X - c\mathcal{L}U) \\ f_U^* &= \beta(\text{id}) \oplus \text{id}.\end{aligned}$$

Clearly: If (Y, f) is a strongly \mathcal{P} -disconnected expansion of X , then $(Y, f) \geq (X_U^*, f_U^*)$ (for all $U \in \mathcal{U}$)

Define $(p_1^* X, p_1^*) = \sup\{(X_U^*, f_U^*): U \in \mathcal{U}\}$. X is strongly \mathcal{P} -disconnected iff $(X, \text{id}) = (p_1^* X, p_1^*)$. The rest can be copied from part (i).

(iii) the same proof as in (ii), but restrict the proof to the set $\tilde{\mathcal{U}} = \{U: U \text{ open and dense in } X \text{ and } U \text{ has } \mathcal{P} \text{ in } X\}$.

(iv) We only show the proof for the set $\mathcal{P}(X)$ of \mathcal{P} -disconnected expansions of X . $(\mathcal{P}X, \leq)$ has a smallest element (pX, p) and a largest element (EX, π) . If $\{(Y_i, f_i)\}_i \subset \mathcal{P}X$ and (Y, f) is the supremum of this set in $\mathcal{E}(X)$, then $(pY, f \circ p)$ is the supremum of this set in $\mathcal{P}X$.

2.5 Remark. Observe that in the previous theorem 2.3 we defined three types of coreflections from the category of Tychonoff spaces and perfect irreducible maps to the category of (strongly) \mathcal{P} -disconnected (quasi- \mathcal{P}) spaces and perfect irreducible maps, for each expansion property \mathcal{P} .

2.6 Proposition. *Let \mathcal{P} be an expansion property and let X be a locally compact space. If X has a \mathcal{P} -disconnected compactification αX , then X has a smallest \mathcal{P} -disconnected compactification $\gamma_{\mathcal{P}}X$. (Similar statements can be made for strongly \mathcal{P} -disconnected compactifications and quasi- \mathcal{P} compactification.)*

Proof. Consider $\tau X = X \cup \{\infty\}$, the one-point compactification of X . The identity map $\text{id}: X \rightarrow X$ can be extended to be perfect irreducible map $f: \alpha X \rightarrow \tau X$. Consider the expansion $(p(\tau X), p)$ of τX . Since αX is \mathcal{P} -disconnected, there is a map $g: \alpha X \rightarrow p(\tau X)$ such that $p \circ g = f$. This implies that $|p^{-1}(x)| = 1$, for all $x \in X \subset \tau X$, i.e. $p(\tau X)$ can be considered as a compactification of X with embedding $(p \wedge p^{-1}(X))^{-1}: X \rightarrow p(\tau X)$. Since every \mathcal{P} -disconnected compactification of X can be mapped onto τX by a perfect and irreducible map, it follows that $p(\tau X)$ is the smallest \mathcal{P} -disconnected compactification of X .

2.7 Remark. Example 2.4(ii) shows that the local compactness of X is essential in the previous proposition. Indeed, if X is not locally compact then X has no smallest compactification. Observe that X is quasi-F (basically disconnected) iff βX is quasi-F (basically disconnected), therefore, each locally compact quasi-F (basically disconnected) space has a smallest quasi-F (basically disconnected) compactification.

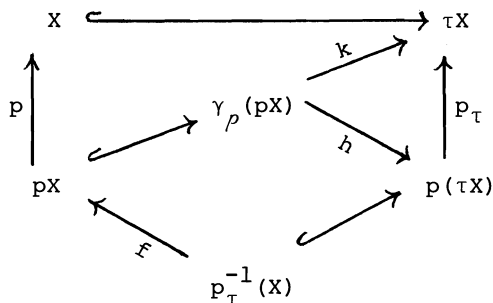
2.8 Proposition. Let \mathcal{P} be an expansion property satisfying the following properties.

(i) X is \mathcal{P} -disconnected, then X has a \mathcal{P} -disconnected compactification.

(ii) Subsets of \mathcal{P} -disconnected spaces which are open and dense are \mathcal{P} -disconnected.

Then if X is locally compact then $p(\tau X) = \gamma_{\mathcal{P}}(pX)$. (Here τX denotes the one point compactification of X .) (Similarly $p^*(\tau X) = \gamma_{\mathcal{P}^*}(p^*X)$ and $\tilde{p}(\tau X) = \gamma_{\tilde{\mathcal{P}}}(\tilde{p}X)$, if \mathcal{P} satisfies (i) and (ii) for the adapted notions.)

Proof. According to (i) and 2.6 each locally compact \mathcal{P} -disconnected space has a smallest \mathcal{P} -disconnected compactification $\gamma_{\mathcal{P}}X$. Consider $p_{\tau}: p(\tau X) \rightarrow \tau X$. Clearly $p_{\tau}^{-1}(X)$ is locally compact and has a \mathcal{P} -disconnected compactification. Obviously, $p(\tau X) = \gamma_{\mathcal{P}}(p_{\tau}^{-1}(X))$. Also (ii) implies that $p_{\tau}^{-1}(X)$ is \mathcal{P} -disconnected, therefore, there is a map $f: p_{\tau}^{-1}(X) \rightarrow pX$ such that $p \circ f = p_{\tau}/\tau^{-1}(X)$. But pX is \mathcal{P} -disconnected and locally compact, hence $\gamma_{\mathcal{P}}(pX)$ exists and clearly there is a perfect irreducible map $k: \gamma_{\mathcal{P}}(pX) \rightarrow \tau X$ which extends $p: pX \rightarrow X$.



Therefore there exists a map $h: \gamma_\rho(pX) \rightarrow p(\tau X)$ such that $k = p_\tau \circ h$. Obviously $h(pX) = p_\tau^{-1}(X)$ and therefore $h = f_e^{-1}$.

Both h and f are homeomorphisms, so we conclude:

$$p(\tau X) = \gamma_\rho(p_\tau^{-1}(X)) = \gamma_\rho(pX).$$

Observe that we also proved that the projective expansion $(p_\tau^{-1}(X), p_\tau)$ is equivalent to (pX, p) .

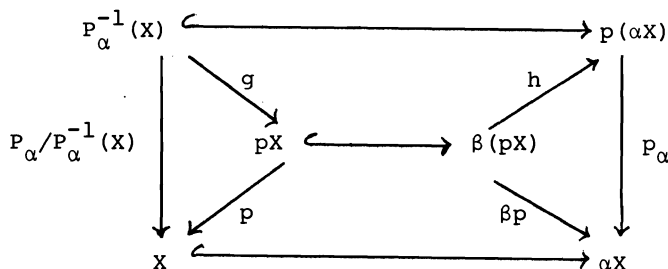
2.9 Proposition. *Let \mathcal{P} be an expansion property satisfying the following.*

(i) *X is \mathcal{P} -disconnected then βX is \mathcal{P} -disconnected.*

(ii) *Dense subsets of \mathcal{P} -disconnected spaces are \mathcal{P} -disconnected.*

Then, if αX is a compactification of X then $(pX, p) = (p_\alpha^{-1}(X), p_\alpha \wedge p_\alpha^{-1}(X))$. (Here $(p(\alpha X), p_\alpha)$ denotes the smallest \mathcal{P} -disconnected expansions of αX .) Similar statements hold for quasi- \mathcal{P} spaces and strongly \mathcal{P} -disconnected spaces.

Proof. Consider $p_\alpha: p(\alpha X) \rightarrow \alpha X$. According to (ii), $p_\alpha^{-1}(X)$ is \mathcal{P} -disconnected, so there is a map $g: p_\alpha^{-1}(X) \rightarrow pX$ such that $g \circ p = p_\alpha \wedge p_\alpha^{-1}(X)$. Consider $p: pX \rightarrow X (\subset \alpha X)$ and its Čech-Stone extension $\beta p: \beta(pX) \rightarrow \alpha X$. Since $\beta(pX)$ is \mathcal{P} -disconnected there is a map $h: \beta(pX) \rightarrow p(\alpha X)$ such that $p_\alpha \circ h = \beta p$.



Then clearly $g: p_\alpha^{-1}(X) \rightarrow pX$ must be injective, and the conclusion follows.

2.10 *Remark.* Propositions 2.7-2.9 were the only satisfying link I could find between expansions of extensions and extensions of expansions. To find such a link is in itself an interesting problem and therefore I would like to ask the following question.

(i) "Can one (or, under what conditions can one) define for a given expansion property \mathcal{P} (with coreflections pX) an associated extension property \mathcal{Q} (with epi-reflection gX) such that $g(pX) = p(gX)$?" (And vice versa.)

A class closely related to the class of basically disconnected spaces and the class of the quasi-F spaces are the so-called F-spaces. Recall that a space is called F if each cozeroset is C^* -embedded. Our final goal is to show that there exists no coreflection of the category of Tychonoff spaces and perfect irreducible maps into the category of F-spaces and perfect irreducible maps. To establish this result it suffices to present an example of a space X with no smallest F-expansion.

2.11 *Example.* Let X be the space obtained from $\omega_1 + 1$ and $\omega_0 + 1$ by identifying the points ω_1 and ω_0 to one point ω . Let $f: \omega_1 + 1 \oplus \omega_0 + 1 \rightarrow X$ be the identification map. We claim that among the expansions of X which are F -spaces there is no smallest. To see this, let us first consider X . Since $\omega_0 \subset X$ is a cozero set of X , $\Lambda^{-1}(\omega_0)$ is a cozero-set of ΛX and therefore $\sigma\mathbb{Z}\Lambda^{-1}(\omega_0)$ is clopen. It follows that $\Lambda X = \Lambda(\omega_1 + 1) \oplus \Lambda(\omega_0 + 1)$.

Let $\Lambda_1: \Lambda(\omega_1 + 1) \rightarrow \omega_1 + 1$ and $\Lambda_0: \Lambda(\omega_0 + 1) \rightarrow \omega_0 + 1$ denote the corresponding expansion maps. Observe that $\Lambda(\omega_0 + 1) \cong \beta\omega_0$ (since the countable discrete set $\Lambda^{-1}(\omega_0)$ is C^* -embedded).

Since ω_1 is a P -point in $\omega_1 + 1$ it follows easily that $|\Lambda_1^{-1}\{\omega_1\}| = 1$. (Indeed, $\Lambda_1^{-1}(\{\omega_1\})$ is a compact P -set in the basically disconnected space $\Lambda(\omega_1 + 1)$. If a compact P -set in a basically disconnected space is identified to one point then the obtained space is basically disconnected.) It follows that $\Lambda(\omega_1 + 1)$ is homeomorphic to the one point compactification of the space of (fixed or non-fixed) non-uniform ultrafilters of the discrete space of cardinality ω_1 .

Let p denote the unique point in $\Lambda_1^{-1}\{\omega_1\}$. Let $x \in \Lambda(\omega_0 + 1) - \Lambda_0^{-1}(\omega_0)$ ($= \beta\omega_0 - \omega_0$). Since p is a P -point, it is easy to see that the space obtained by identifying p and x is an F -space (and of course, it is an expansion of X). Therefore, if there would exist a smallest element (\tilde{X}, g) among the F -expansions, all the points of $\Lambda(\omega_0 + 1) - \Lambda_0^{-1}(\omega_0)$ must be identified with p in \tilde{X} , i.e. $|g^{-1}(\omega)| = 1$.

But then the cozeroset $g^{-1}(\omega_0)$ is no longer C^* -embedded in \tilde{X} . Therefore no such smallest element exists.

It is interesting to observe that, although there are no minimum F -expansions of X , there exist minimal F -expansions of X . (Namely the ones obtained from $\Lambda(\omega_1 + 1)$ and $\Lambda(\omega_0 + 1)$ by identifying p with some $x \in \Lambda(\omega_0 + 1) - \Lambda_0^{-1}(\omega_0)$.) I do not know whether these minimal F -expansions exist for arbitrarily compact spaces. Observe that the smallest quasi- F expansion of X can be obtained from $\Lambda(\omega_1 + 1) \oplus \Lambda(\omega_0 + 1)$ by identifying p and $\Lambda(\omega_0 + 1) - \Lambda_0^{-1}(\omega_0)$ to one point.

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