TOPOLOGY PROCEEDINGS
Volume 9, 1984
Pages 193–196

http://topology.auburn.edu/tp/

Research Announcement:

CONTINUA OF CONSTANT DISTANCES RELATED TO THE SPANS

by

A. Lelek

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Research Announcement

CONTINUA OF CONSTANT DISTANCES RELATED TO THE SPANS

A. Lelek

The purpose of this note is to announce some results which complement and can, perhaps, offer a better handling of the concept of the span for compact metric spaces. A complete version with proofs and applications will be published elsewhere.

All spaces are assumed to be non-empty metric spaces, and all mappings to be continuous functions. Let $f: X \to Y$ be a mapping. If X is connected, the *surjective span* $\sigma^*(f)$ of f is defined to be the least upper bound of the set of real numbers α with the following property: there exist non-empty connected sets $C_\alpha \subset X \times X$ such that $\text{dist}[f(x), f(x')] \geq \alpha$ for $(x,x') \in C_\alpha$, and

$$(\sigma^*) \qquad p_1(C_\alpha) = p_2(C_\alpha) = X,$$

where p_1 and p_2 denote the standard projections of the product, that is, $p_1(x,x') = x$ and $p_2(x,x') = x'$. The *span* $\sigma(f)$, the *semispan* $\sigma_0(f)$, both for mappings f with the domains X not necessarily connected, and the *surjective semispan* $\sigma_0^*(f)$ in the case of connected domains, are defined similarly with condition (σ^*) relaxed to conditions

$$(\sigma) \qquad p_1(C_\alpha) = p_2(C_\alpha),$$

$$(\sigma_0) \qquad p_1(C_\alpha) \supset p_2(C_\alpha),$$

$$(\sigma_0^*) \qquad p_1(C_\alpha) = X,$$

respectively. The following inequalities are direct consequences of the definitions:

$$0 \leq \sigma^*(f) \leq \sigma(f) \leq \sigma_0(f) \leq \operatorname{diam} Y,$$

$$0 \leq \sigma^*(f) \leq \sigma_0^*(f) \leq \sigma_0(f) \leq \operatorname{diam} Y.$$

For $\tau = \sigma, \sigma^*, \sigma_0, \sigma_0^*$, the corresponding spans $\tau(X)$ of a space X are the spans $\tau(\mathrm{id}_X)$ of the identity mapping on X. The span $\sigma(f)$ of a mapping f was originally defined by Ingram [1], while the present author earlier introduced the span $\sigma(X)$ and, subsequently, the other types of these quantities for metric spaces (see [2] and [3]). It is known that, for some particular spaces, neither two of these four types of spans need to be equal (see [3] and [4]).

One of the consequences of our results is that, in the definition of the span $\sigma(f)$ for compact domains, the inequality $\operatorname{dist}[f(x), f(x')] \geq \alpha$ can be replaced by the equality. The author wishes to thank M. B. de Castro and L. G. Oversteegen for helpful discussions concerning such a possibility. It can be derived directly from the theorem below, and the same replacement can also be made in the definitions of other types of spans. The two kinds of definitions, one using the equality and another one using the inequality, are then equivalent, respectively, in each of the four cases of the concepts involved for mappings of compact metric spaces. By a *continuum* we understand a connected compact metric space.

Theorem. If $f: X \to Y$ *is a mapping,* X *is a compact metric space,* $\tau = \sigma, \sigma_0$, *and* $0 \leq \beta \leq \tau(f)$, *then there exists a non-empty continuum* $K_\beta \subset X \times X$ *such that*

$$\operatorname{dist}[f(x), f(x')] = \beta$$

for $(x,x') \in K_\beta$, *and condition* (τ) *is satisfied for* K_β *in lieu of* C_α, *respectively. Moreover, if* X *is a continuum, the same conclusion also holds for* $\tau = \sigma^*, \sigma_0^*$.

The scheme of the proof is illustrated, rather vaguely, in the following diagram:

$$\begin{tikzcd}
U_n \arrow[d, "h_n"'] & & \\
P_n \arrow[drr, "\overline{f}|P_n"] & & \\
X \arrow[u, "g_n"] \arrow[rr, "f"] & & f(X) \subset I^\omega \\
\cap & & \\
R^\omega \arrow[uurr, "\overline{f}"'] & &
\end{tikzcd}$$

The space X is considered embedded in the Hilbert space R^ω, and its image f(X) is considered embedded in the Hilbert cube I^ω. The mapping f is extended over R^ω to a mapping $\overline{f}$. For each $n = 1,2,\cdots$, a polyhedron P_n is taken in the (1/n)-neighborhood of X in R^ω such that there exists a (1/n)-translation g_n of X into P_n. The space U_n is the universal covering space of P_n with h_n the covering projection. We use the unicoherence of the product $U_n \times U_n$ to find a connected set M_n which cuts $U_n \times U_n$ between certain two points and is contained in a properly selected subset of $U_n \times U_n$. The desired continuum K_β is then defined to be the limit of a convergent subsequence of the sequence of the closures of the sets $(h_n \times h_n)(M_n)$ $(n = 1,2,\cdots)$.

References

[1] W. T. Ingram, *An atriodic tree-like continuum with positive span*, Fund. Math. 77 (1972), 99-107.

[2] A. Lelek, *Disjoint mappings and the span of spaces*, Fund. Math. 55 (1964), 199-214.

[3] ________, *An example of a simple triod with surjective span smaller than span*, Pacific J. Math. 64 (1976), 207-215.

[4] ________, *On the surjective span and semispan of connected metric spaces*, Colloq. Math. 37 (1977), 35-45.

University of Houston

Houston, Texas 77004