Research Announcement:

CONTINUA OF CONSTANT DISTANCES RELATED TO THE SPANS

by

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The purpose of this note is to announce some results
which complement and can, perhaps, offer a better handling
of the concept of the span for compact metric spaces. A
complete version with proofs and applications will be pub-
lished elsewhere.

All spaces are assumed to be non-empty metric spaces,
and all mappings to be continuous functions. Let $f: X \to Y$
be a mapping. If X is connected, the *surjective span*
$\sigma^*(f)$ of f is defined to be the least upper bound of the
set of real numbers α with the following property: there
exist non-empty connected sets $C_\alpha \subset X \times X$ such that
$\text{dist}[f(x), f(x')] \geq \alpha$ for $(x,x') \in C_\alpha$, and

$(\sigma^*) \qquad p_1(C_\alpha) = p_2(C_\alpha) = X,$

where p_1 and p_2 denote the standard projections of the
product, that is, $p_1(x,x') = x$ and $p_2(x,x') = x'$. The
span $\sigma(f)$, the *semispan* $\sigma_0(f)$, both for mappings f with
the domains X not necessarily connected, and the *surjective
semispan* $\sigma_0^*(f)$ in the case of connected domains, are defined
similarly with condition (σ^*) relaxed to conditions

$(\sigma) \qquad p_1(C_\alpha) = p_2(C_\alpha),$

$(\sigma_0) \qquad p_1(C_\alpha) \supset p_2(C_\alpha),$

$(\sigma_0^*) \qquad p_1(C_\alpha) = X,$

respectively. The following inequalities are direct
consequences of the definitions:

$$0 \leq \sigma^*(f) \leq \sigma(f) \leq \sigma_0(f) \leq \text{diam } Y,$$

$$0 \leq \sigma^*(f) \leq \sigma_0^*(f) \leq \sigma_0(f) \leq \text{diam } Y.$$

For $\tau = \sigma, \sigma^*, \sigma_0, \sigma_0^*$, the corresponding spans $\tau(X)$ of a space X are the spans $\tau(\text{id}_X)$ of the identity mapping on X. The span $\sigma(f)$ of a mapping f was originally defined by Ingram [1], while the present author earlier introduced the span $\sigma(X)$ and, subsequently, the other types of these quantities for metric spaces (see [2] and [3]). It is known that, for some particular spaces, neither two of these four types of spans need to be equal (see [3] and [4]).
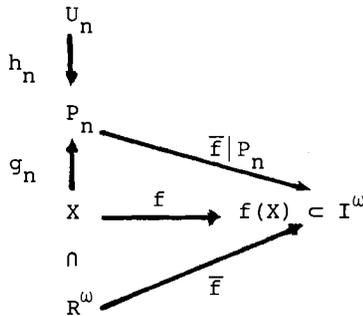
One of the consequences of our results is that, in the definition of the span $\sigma(f)$ for compact domains, the inequality dist$[f(x), f(x')] \geq \alpha$ can be replaced by the equality. The author wishes to thank M. B. de Castro and L. G. Oversteegen for helpful discussions concerning such a possibility. It can be derived directly from the theorem below, and the same replacement can also be made in the definitions of other types of spans. The two kinds of definitions, one using the equality and another one using the inequality, are then equivalent, respectively, in each of the four cases of the concepts involved for mappings of compact metric spaces. By a *continuum* we understand a connected compact metric space.

Theorem. If f: X → Y *is a mapping,* X *is a compact metric space,* $\tau = \sigma$, σ_0, *and* $0 \leq \beta \leq \tau(f)$, *then there exists a non-empty continuum* $K_\beta \subset X \times X$ *such that*

$$\text{dist}[f(x), f(x')] = \beta$$

for $(x,x') \in K_\beta$, *and condition* (τ) *is satisfied for* K_β *in lieu of* C_α, *respectively. Moreover, if* X *is a continuum, the same conclusion also holds for* $\tau = \sigma^*, \sigma_0^*$.

The scheme of the proof is illustrated, rather vaguely, in the following diagram:

$$
\begin{array}{c}
U_n \\
h_n \downarrow \\
P_n \\
g_n \uparrow \\
X \xrightarrow{\ f\ } f(X) \subset I^\omega \\
\cap \\
R^\omega
\end{array}
$$

with arrows labeled $\overline{f}|P_n$ and $\overline{f}$ pointing to $f(X) \subset I^\omega$.

The space X is considered embedded in the Hilbert space R^ω, and its image f(X) is considered embedded in the Hilbert cube I^ω. The mapping f is extended over R^ω to a mapping $\overline{f}$. For each $n = 1,2,\cdots$, a polyhedron P_n is taken in the (1/n)-neighborhood of X in R^ω such that there exists a (1/n)-translation g_n of X into P_n. The space U_n is the universal covering space of P_n with h_n the covering projection. We use the unicoherence of the product $U_n \times U_n$ to find a connected set M_n which cuts $U_n \times U_n$ between certain two points and is contained in a properly selected subset of $U_n \times U_n$. The desired continuum K_β is then defined to be the limit of a convergent subsequence of the sequence of the closures of the sets $(h_n \times h_n)(M_n)$ $(n = 1,2,\cdots)$.

References

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[2] A. Lelek, *Disjoint mappings and the span of spaces*, Fund. Math. 55 (1964), 199-214.

[3] _____, *An example of a simple triod with surjective span smaller than span*, Pacific J. Math. 64 (1976), 207-215.

[4] _____, *On the surjective span and semispan of connected metric spaces*, Colloq. Math. 37 (1977), 35-45.

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