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A METRIC FOR METRIZABLE GO-SPACES

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Conditions which force the metrizability of GO-spaces are well known (see [Fa]). Since GO-spaces are T_3-spaces and countable GO-spaces are first countable it follows that countable GO-spaces are metrizable. However it is not always apparent what a metric is for a given metrizable G)-space even if it is countable. For example the Sorgenfry line [S] restricted to the set of rational numbers or, if $\alpha < \omega_1$, the LOTS $[0,\alpha]$ are both countable and, thus, metrizable but it is difficult to construct a metric for either of these spaces ([A]). In this note a metric is derived for GO-spaces.

A LOTS (= linearly ordered topological space) is a triple $(X,\lambda(\leq),\leq)$ where $(X,\leq)$ is a linearly ordered set and $\lambda(\leq)$ is the usual open-interval topology generated by the order $\leq$.

Recall that a subset A of X is order-convex if whenever a and b are in A, then each point lying between a and b is also in A.

A GO-space (= generalized ordered space) is a subspace of a LOTS (see [L]). There is an equivalent way to obtain a GO-space X by starting with a linearly-ordered set Y. Equip Y with a topology τ that contains $\lambda(\leq)$ and has a base of open sets each of which is order-convex. In this case X is said to be constructed on Y and $X = GO_Y(R,E,I,L)$ where

$$I = \{x \in X | \{x\} \in \tau\},$$

$$R = \{x \in X - I | [x,\rightarrow[\in \tau\},$$

$$L = \{x \in X - I |]\leftarrow,x] \in \tau\}, \text{ and}$$

$$E = X - (R \cup I \cup L).$$

See [L] for further notation.

In deriving a metric for a metrizable GO-space it is illuminating to first derive a metric for a countable GO-space case and then derive the metric for an arbitrary metrizable GO-space.

Let Q denote the LOTS of rational numbers and let N denote the set of natural numbers. An order $\leq$ on a set X is a dense-order if whenever $a,b \in X$ are such that $a < b$ then there is a point c in X such that $a < c < b$.

The next theorem indicates how a GO-space may be embedded in a LOTS.

Theorem 1. *If* $X = GO_Y(R,E,I,L)$, *then* X *is homeomorphic to a subspace of a dense-ordered* LOTS L(X). *Furthermore the homeomorphism is order-preserving and* L(X) *does not have any endpoints.*

Proof. Let

$$\begin{aligned} L(X) = &\{(x,q) | x \in I, q \in]-1,1[\cap Q\} \cup \\ &\{(x,q) | x \in R, q \in]-1,0] \cap Q\} \cup \\ &\{(x,q) | x \in L, q \in [0,1[\cap Q\} \cup \\ &\{(x,0) | x \in E\}. \end{aligned}$$

Equip L(X) with the lexicographic ordering induced from the order on X and the natural order on Q. It follows that L(X) is a dense-ordered LOTS without endpoints. Define a function ϕ from X into L(X) by $\phi(x) = (x,0)$. Then ϕ is an order-preserving homeomorphism from X into L(X).

Corollary. A countable GO-*space* X *is homeomorphic to a subspace of* Q *by an order-preserving homeomorphism.*

Proof. Since X is countable, L(X) is a countable, dense-ordered LOTS without endpoints. Hence L(X) is homeomorphic to Q by an order-preserving homeomorphism [Fr]. Thus X is homeomorphic to a subspace of Q by an order-preserving homeomorphism.

Since each countable GO-space X can be considered a subspace of Q the usual metric on Q restricted to X is a metric on X. Unfortunately it is often difficult to use this metric since it is hard to visualize the embedding.

Let X be a countable GO-space and ϕ be an order-preserving homeomorphism from X into L(X) and β be an order-preserving homeomorphism from L(X) onto Q. Notice that if $x \in R$ ($x \in L$) then there is an interval J in L(X) immediately preceding (succeeding) $\phi(x)$ such that no point of X maps into $\beta(J)$ and if $x \in I$ then there are intervals J_1 and J_2 in L(X) such that J_1 immediately precedes $\phi(x)$ and J_2 immediately succeeds $\phi(x)$ and no point of X maps into $\beta(J_1 \cup J_2)$. By considering Q homeomorphic to $Q \cap]0,1[$ and embedding X in $Q \cap]0,1[$ it follows that the image of those intervals in $Q \cap]0,1[$ must be made arbitrarily small if $|R \cup L \cup I|$ is large.

Let $\mathcal{K}$ be the collection of all maximal, nondegenerate, convex subsets of $X - (R \cup L \cup I)$. Then $\mathcal{K}$ is at most countable. Let $K_1, K_2, \cdots$ be an enumeration of $\mathcal{K}$ (without repetitions). Since each K_i is homeomorphic to a convex subset of Q it is metrizable. Let d_i be a metric for K_i

that is bounded by 1. Let $x_1, x_2, \cdots$ be a counting of $R \cup L \cup I$.

These observations motivate the derivation of a metric for a countable GO-space.

To define the function on $X \times X$ compensation functions must be defined for the points of X. The motivation for these compensation functions comes from observing how X is embedded in Q and how one would "travel" in Q from point to point. Let

$$\phi_\ell(x) = \begin{cases} 2^{-n} & \text{if } x = x_n \in R \cup I \\ 0 & \text{if } x \in L \cup E \end{cases}$$

and

$$\phi_r(x) = \begin{cases} 2^{-n} & \text{if } x = x_n \in L \cup I \\ 0 & \text{if } x \in R \cup E. \end{cases}$$

A metric function σ can be defined on $X \times X$. Although it is not necessary it is convenient to consider cases. Let $a < b$.

Case 1. If $\{a,b\} \subseteq \cup\mathcal{K}$ and both lie in the same K_i let

$$\sigma(a,b) = d_i(a,b) \cdot 2^{-1}$$

and if $a \in K_i$ and $b \in K_j$ for $i \neq j$, then let

$$\begin{aligned}\sigma(a,b) = {} & \operatorname{Sup}\{2^{-1} \cdot d_i(a,z) \mid a < z, z \in K_i\} + \\ & \Sigma\{2^{-n} \mid K_n \subset \,]a,b[\} + \\ & \Sigma\{\sigma_\ell(x) + \phi_r(x) \mid a < x < b\} + \\ & \operatorname{Sup}\{2^{-j} \cdot d_j(z,b) \mid z < b, z \in K_j\}.\end{aligned}$$

Case 2. If $a \in K_j$ and $b \notin \cup\mathcal{K}$ then let

$$\begin{aligned}\sigma(a,b) = {} & \operatorname{Sup}\{2^{-j} \cdot d_j(a,z) \mid a < z, z \in K_j\} + \\ & \Sigma\{2^{-n} \mid K_n \subset \,]a,b[\} + \\ & \Sigma\{\phi_\ell(x) + \phi_r(x) \mid a < x < b\} + \\ & \phi_\ell(b).\end{aligned}$$

If $a \notin \cup\mathcal{K}$ and $b \in K_j$, then let

$$\sigma(a,b) = \phi_r(a) + \Sigma\{2^{-n} | K_n \subset]a,b[\} + \Sigma\{\phi_\ell(x) + \phi_r(x) | a < x < b\} + \operatorname{Sup}\{2^{-1} \cdot d_i(z,b) | z < b, z \in K_i\}.$$

Case 3. If neither a nor b is in $\cup\mathcal{K}$, then let

$$\sigma(a,b) = \phi_r(a) + \Sigma\{2^{-n} | K_n \subset]a,b[\} + \Sigma\{\phi_\ell(x) + \phi_r(x) | a < x < b\} + \phi_\ell(b).$$

Furthermore let $\sigma(a,b) = 0$ if and only if $a = b$ and let $\sigma(a,b) = \sigma(b,a)$ for a and b in X.

Theorem 1. *If* X *is a countable* GO-*space, then* σ *is a metric on* X.

Proof. Since each of the series used in defining σ is bounded by the convergent series $2 \cdot \Sigma 2^{-n}$, it follows that σ is well-defined. Since σ was constructed to be a metric function it is just a matter of cases to check that σ defines the topology. Let $S_\sigma(x,\varepsilon)$ denote the sphere centered at x whose σ-radius is ε.

Case 1. If $x_n \in I$, then $S_\sigma(x_n, 2^{-n}) = \{x_n\}$.

Case 2. Let $x_n \in R$ and $S_\sigma(x_n,\varepsilon)$ be given. Since $x_n \in R$ choose $x \in X$ such that $x_n < x$ and $x \in S_\sigma(x_n,\varepsilon)$. Then $[x_n, x[\subset S_\sigma(x_n,\varepsilon)$.

Let $[x_n, x[$ be given. If $\sigma(x_n,x) = \varepsilon_1$, let $\varepsilon = \min\{2^{-n}, \varepsilon_1\}$. Then $[x_n, x[\supset S_\sigma(x_n,\varepsilon)$.

Case 3. If $x_n \in L$ argue analogously to Case 2.

Case 4. If $x_n \in E$ then argue on each side of x_n using Case 2 and Case 3.

Hence σ is a metric for X.

If $R \cup L \cup I$ is dense in X then the metric σ is much less simple as the next two examples illustrate.

Example 1. Let X be the Sorgenfrey Line restricted to Q, that is, $GO_Q(Q,\Phi,\Phi,\Phi)$. Let $q_1, q_2, \cdots$ be a counting of Q. Then, for each $k \in N$, $\phi_\ell(q_k) = 2^{-k}$ and $\phi_r(q_k) = 0$. Thus if $q_n < q_m$, it follows that

$$\sigma(q_n, q_m) = \Sigma\{2^{-k} | q_n < q_k \leq q_m\}.$$

Example 2. Let X be the LOTS $[1,\alpha)$ where $\alpha < \omega_1$. Then $R = E = \Phi$. Let $x_1, x_2, \cdots$ be a counting of $[1,\alpha)$. Then if x_k is a non-limit ordinal $\phi_r(x_k) = \phi_\ell(x_k) = 2^{-k}$ and if x_k is a limit ordinal $\phi_\ell(x_k) = 0$ and $\phi_r(x_n) = 2^{-n}$. Hence, if $x_n < x_m$, then

$$\sigma(x_n, x_m) = \phi_r(x_n) + \Sigma\{\phi_r(x) + \phi_\ell(x) | x_n < x < x_m\} + \phi_\ell(x_m).$$

The following corollary easily follows.

Corollary. If $R \cup L \cup I$ *is dense in the countable* GO-*space* X *then*

$$\sigma(a,b) = \phi_r(a) + \Sigma\{\phi_r(x) + \phi_\ell(x) | a < x < b\} + \phi_\ell(b)$$

is a metric for X.

The countable GO-space case motivates the metrizable GO-space case by realizing the countable GO-spaces are σ-discrete.

The following theorem gives structural conditions for the metrizability of a given GO-space.

Theorem 2 [Fa]. *Let* X *be a* GO-*space. The following properties are equivalent.*

(i) X *is metrizable, and*

(ii) *There is a dense, σ-discrete set* D *in* X *containing* R ∪ L.

It follows from this result that if X is a metrizable GO-space then each of R and L are σ-discrete in X. Since I is open in X it is an F_σ-set and, hence, a σ-discrete set.

Let $R = \cup\{R_n | n = 1,2,\cdots\}$, $L = \cup\{L_n | n = 1,2,\cdots\}$ and $I = \cup\{I_n | n = 1,2,\cdots\}$ where for each n, $R_n \subseteq R_{n+1}$, $L_n \subseteq L_{n+1}$ and $I_n \subseteq I_{n+1}$.

If $X = GO_Y(R,E,I,L)$ is a metrizable GO-space where Y is a metric LOTS with metric d then in order to find a metric for X compensation functions must be found (as in the countable case). This is motivated by embedding X in L(X) and observing how one "travels" from point to point. If $x \leq y$, let $R(x,y) = 2^{-i}$, where i is the first natural number such that $R_i \cap \,]x,y] \neq \emptyset$. If no such i exists let $R(x,y) = 0$. Let $L(x,y) = 2^{-j}$ where j is the first natural number such that $L_j \cap [x,y[\, \neq \emptyset$. If no such j exists let $L(x,y) = 0$. If $x < y$ let $I(x,y) = 2^{-k}$ where k is the first natural number such that $I_K \cap [x,y] \neq \Phi$. If no such k exists or if $x = y$ let $I(x,y) = 0$.

Let

$$\rho(y,x) = \rho(x,y) = d(x,y) + R(x,y) + L(x,y) + I(x,y).$$

It is a matter of checking cases to see that ρ is a metric function on X. Notice if $y_1 < y_2$ and $x < y_1$ then $\rho(x,y_1) \leq \rho(x,y_2)$.

Theorem 2. *Let* Y *be a* LOTS *with metric* d *and* $X = GO_Y(R,E,I,L)$ *be a metrizable* G)*-space. Then* ρ*, defined above, is a metric on* X.

Proof. All that needs to be shown is that ρ preserves the topology on X. Consider the following cases:

(i) If $x \in I$ then let k be the first natural number such that $x \notin I_K$. It follows that $S_\rho(x,2^{-k}) = \{x\}$.

(ii) If $x \in R$ and $S_\rho(x,\varepsilon)$ is given, choose the first natural number n such that $3 \cdot 2^{-n} < \varepsilon \cdot 2^{-2}$. Let $K_n = \cup\{R_i \cup L_i \cup I_i | i = 1,\cdots,n\}$. Choose $y > x$ such that $d(x,y) < \varepsilon \cdot 2^{-2}$ and $]x,y[\cap K_n = \Phi$. This can be done since K_n is discrete and $x \in R$. It follows that $\rho(x,y) < \varepsilon \cdot 2^{-2} + 3 \cdot \varepsilon \cdot 2^{-2} = \varepsilon$. Thus $[x,y[\subset S_\rho(x,\varepsilon)$.

If $[x,b[$ is given let n be the first natural number such that $x \in R_n$. Let $\varepsilon = \min\{d(x,b),2^{-n}\}$. Then $S_\rho(x,\varepsilon) \subseteq [x,b[$.

(iii) If $x \in L$ argue analogously to (ii).

(iv) If $x \in E$ combine (ii) and (iii).

Hence ρ preserves the topology on X and, hence, is a metric for X.

Corollary. If R ∪ L ∪ I *is dense in the metrizable* GO-*space* X *then*

$$\rho(x,y) = R(x,y) + L(x,y) + I(x,y)$$

is a metric on X.

Let E denote the real line with the usual order topology.

Example 3. Let $X = GO_E(Q, E-Q, \Phi, \Phi)$ and let $q_1, q_2, \cdots$ be any counting of the rational numbers. Then

$$\rho(x,y) = R(x,y) = 2^{-j}$$

(where q_j is the first rational number in $]x,y]$) is a metric on X.

If $Y = GO_Q(Q, \Phi, \Phi, \Phi)$ (i.e., the Sorgenfrey Line) then the above ρ is a metric on Y that is simpler than the metric given in Example 1.

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