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by

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BETWEEN MINIMAL HAUSDORFF AND COMPACT HAUSDORFF SPACES

R. F. Dickman, Jr. and J. R. Porter[1]

1. Introduction and Preliminaries

There are three well-known classes of spaces which lie strictly between the class of compact Hausdorff spaces and the class of minimal Hausdorff spaces--functionally compact spaces, C-compact spaces, and seminormal, H-closed spaces; these spaces have been investigated by various authors [DP, DZ, FP, GV_1, GV_2, LT, S, V_1, V_2, W]. A compact Hausdorff space is seminormal and H-closed, a seminormal, H-closed space is C-compact, and a functionally compact space is minimal Hausdorff; none of these implications can be reversed. In Section 3 of this paper we answer the embedding problem for C-compact spaces (see [DZ, GV_2]) by giving an example of an H-closed space (and an example of a seminormal space) which can not be embedded in a C-compact space. Also, a problem by Lim and Tan [LT] is solved. In Section 2, two new concepts, FFC and CFC spaces, are introduced and developed. A functionally compact space is CFC, a CFC space is FFC, and a FFC space is minimal Hausdorff; examples are given to show that these implications can not be reversed. Surprisingly, the theory of FFC spaces is quite similar to the theory of minimal Hausdorff spaces. Each Hausdorff space is embeddable in some FFC space and an

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example of an FFC space which is not CFC is shown to answer a problem by Vermeer [Ve]. Some questions are listed in Section 4.

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We now introduce some definitions and state some preliminary results which are needed in the sequel. *All spaces considered in this paper are Hausdorff.*

A space X is H-closed (abbreviated as HC) if X is closed whenever it is a subspace of another space. A set A in X is *regular open* if $A = int_X cl_X A$. The collection of all regular open subsets of X is closed under finite intersections and is a base for a topology on the underlying set of X; X with this topology is denoted as X_s. A space X is said to be *semiregular* if $X = X_s$. Clearly, the identity function from X to X_s is continuous, and it is easy to show that the space X_s is semiregular. A subset A of a space X has a *base of regular open sets* in X if for each open set U containing A, there is a regular open set V such that $A \subseteq V \subseteq U$. In particular, a space is semiregular iff each singleton has a base of regular open sets. A subset A of a space X is *regular closed* if X\A is regular open, in particular, if $A = cl_X int_X A$. It is known [K] that a regular closed subset of an HC space is an HC subspace.

A function $f: X \to Y$, where X and Y are spaces, is θ-*continuous* if for each $p \in X$ and open set U containing f(p), there is an open set V containing p such that $f[cl_X V] \subseteq cl_Y U$. Clearly, a continuous function is

θ-continuous, and a θ-continuous function into a regular space is continuous.

A subset A of a space X is an H-*set in* X if for each cover $\mathcal{C}$ of A by sets open in X, there is a finite subfamily $\mathcal{F} \subseteq \mathcal{C}$ such that $A \subseteq \cup\{cl_X U: U \in \mathcal{F}\}$. Our next result contains some basic facts about H-sets; most are contained in [DP, PT, Ve, W] and the rest are easy to verify.

(1.1) Let X be a space and $A, B \subseteq X$. The following are true:

(a) If A is an HC subspace, then A is an H-set in X.

(b) If A is regular closed in X, $A \subseteq B$, and B is an H-set in X, then A is an HC subspace.

(c) If A and B are H-sets in X, then so is $A \cup B$.

(d) The set X is an H-set in X iff X is HC.

(e) The subset A is an H-set in X iff for each open filter base $\mathcal{F}$ on X such that $F \cap A \neq \emptyset$ for each $F \in \mathcal{F}$, it follows that $\emptyset \neq ad_X\mathcal{F} \cap A$ ($ad_X\mathcal{F} = \cap\{cl_X F: F \in \mathcal{F}\}$).

(f) If Y is a space, $f: X \to Y$ is a θ-continuous function, and A is an H-set in X, then f[A] is closed and an H-set in Y.

Note that by 1.1(f) an H-set in a space X is closed in X. A subset A is θ-*closed* in a space X if for each point $p \in X\backslash A$, there is an open set U in X such that $p \in U \subseteq cl_X U \subseteq X\backslash A$. In particular, a θ-closed subset is also closed. If A is a subspace of X, then X/A is used to denote the quotient space [which is not necessarily Hausdorff] of X with A identified to a point; we will think of X/A as the set $(X\backslash A) \cup \{\infty\}$ where A is identified as the point ∞ in X/A. The quotient function from X to X/A is denoted as p_A, and q_A denotes $s \circ p_A$ where $s : X/A \to (X/A)_s$ is the identity function. Thus, the function q_A is continuous but, in general, neither X/A nor $(X/A)_s$ is Hausdorff.

(1.2) Let X be a space and $A, B \subseteq X$. The following are true:

(a) If A is θ-closed in X, B is an H-set in X, and $A \subseteq X\backslash B$, there is a regular open set U such that $A \subseteq U \subseteq X\backslash B$.

(b) [DP] The set A is θ-closed in X iff X/A is Hausdorff.

(c) If the function q_A is closed, then A has a local base of regular open sets.

(d) [DP] If Y is a space, $f: X \to Y$ is a θ-continuous function, and $p \in Y$, then $f^{-1}(p)$ is θ-closed in X.

Proof. The proof of (c) is straightforward. So, only the proof of (a) is given. For each $p \in B$, let U_p be an open set such that $p \in U_p \subseteq cl_X U_p \subseteq X\backslash A$. Since B is an H-set in X, there is a finite subset $F \subseteq B$ such that $B \subseteq \cup\{cl_X U_p : p \in F\}$. Let $U = \cup\{U_p : p \in F\}$. Then $A \subseteq X\backslash cl_X U \subseteq X\backslash B$ and $X\backslash cl_X U$ is regular open.

A space is *minimal Hausdorff* (abbreviated as MH) if there is no strictly coarser Hausdorff topology. A space is C-*compact* (abbreviated as CC) if every closed set is an H-set. A space is *seminormal* (resp. θ-*seminormal*) if every closed (resp. θ-closed) set has a base of regular open sets. A space X is *functionally compact* (FC) if each continuous function from X into a space is closed.

(1.3) The following are true for a space X:

(a) [K] The space X is MH iff X is HC and semiregular.

(b) [GV_2] The space X is FC iff X is HC and θ-seminormal.

(c) A space X is MH iff each continuous function from X into a space with singleton point-inverses is closed.

(d) [V_2] A seminormal, HC space is CC.

(e) [DZ, V_2] A CC space is FC.

(f) [DZ] A FC space is MH.

(g) [DP] The space X is FC iff every θ-continuous function from X into a space is closed.

Proof. The proof of (c) is straightforward.

Let X be a space and $\mu X = X \cup \{\mathcal{U}: \mathcal{U}$ is an open ultrafilter on X such that $ad_X \mathcal{U} = \phi\}$. For each open set U in X, let $oU = U \cup \{\mathcal{U} \in \mu X \backslash X: int_X cl_X V \subseteq U$ for some $V \in \mathcal{U}\}$. Most of the parts of the following result is contained in [Po, PV_1, PV_2]; the proof of the other parts are straightforward.

(1.4) Let X be a space. The following are true:

(a) The collection $\{oU: U$ is open in X$\}$ is a base for an HC topology on μX, and μX is an extension of X.

(b) The extension μX is semiregular (and hence, MH by (a) and 1.3(a)) iff X is semiregular.

(c) If V is an open subset of μX, then $int_{\mu X} cl_{\mu X} V = o(int_X cl_X (V \cap X))$ and $cl_{\mu Y}(V \cap X) = cl_{\mu Y} o(V \cap X)$.

(d) If U is a regular open subset of X, then $cl_{\mu X} oU = cl_X U \cup oU$.

(e) If U and V are disjoint open subsets of X, then $oU \cap oV = \phi$.

(f) If X is semiregular, $A \subseteq \mu X \backslash X$, and A is closed in μX, then A is compact.

(g) If $U = \cup\{U_i : 1 \leq i \leq m\}$ where U_i are regular open subsets of Y and $1 \leq i \leq m$, then $oU = \cup\{oU_i : 1 \leq i \leq m\}$.

Let $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of the positive integers $\mathbb{N}$; for $a,b \in \mathbb{N}^*$ where $a < b$ (we define $n < \infty$ for all $n \in \mathbb{N}$), the symbol $[a,b]$ (resp. $[a,b)$) means $\{c \in \mathbb{N}^* : a \leq c \leq b\}$ (resp. $\{c \in \mathbb{N}^* : a \leq c < b\}$). Let X be a space and $\mathcal{S}(X) = X \times \mathbb{N}^*$. A topology on $\mathcal{S}(X)$ is defined as follows: $U \subseteq \mathcal{S}(X)$ is open if whenever $(x,\infty) \in U$ for some $x \in X$, then there is some open set V in X and $n \in \mathbb{N}$ such that $(x,\infty) \in V \times [n,\infty] \subseteq U$. Note that each point of $X \times \mathbb{N}$ is isolated in $\mathcal{S}(X)$. The proof of the next result is straightforward but tedious.

(1.5) Let X be a space and $Y = \mathcal{S}(X)$.

(a) The space Y is seminormal and X is homeomorphic to the subspace $X \times \{\infty\}$ of Y.

(b) If $M \subset \mathbb{N}$ and M and $\mathbb{N}\backslash M$ are infinite sets, then $X \times M$ is a regular open subset of Y; in particular, $X \times \{\infty\}$ is a closed, nowhere dense subset of μY.

(c) If U is an open subset of Y, then $cl_Y U \backslash U \subseteq X \times \{\infty\}$.

(d) If $\{B_i: i \in I\}$ is a family of open subsets of X and $\{n_i: i \in I\}$ is a family of positive integers, then $\cup\{B_i \times [n_i,\infty]: i \in I\}$ is a regular open subset of Y.

We close this section with two examples.

(1.6) *Examples.*

(a) [BPS] Let $X = (\mathbb{N} \times \mathbb{Z}) \cup \{\pm\infty\}$ where $\mathbb{Z}$ is the set of all integers. A set $U \subseteq X$ is defined to be open if $(n,0) \in U$ implies $\{(n,k): |k| \geq m\} \subseteq U$ for some $m \in \mathbb{N}$ and $\infty \in U$ (resp. $-\infty \in U$) implies $[k,\infty) \times [1,\infty) \subseteq U$ (resp. $[k,\infty) \times (-\infty,-1] \subseteq U$) for some $k \in \mathbb{N}$. The space X is MH; X is not FC as the θ-closed set $\{-\infty,\infty\}$ does not have a base of regular open sets.

(b) $[V_1]$ Let $X = (\mathbb{N} \times (\mathbb{N} \cup \{0\})) \cup \{\infty\}$. Suppose $\{\mathbb{N}_i: i \in \mathbb{N}\}$ is a partition of $\mathbb{N}$ into infinite subsets. A set $U \subseteq X$ is defined to be open if $(n,0) \in U$ implies $\{n\} \times [k,\infty) \cup [k,\infty) \times \mathbb{N}_n \subseteq U$ for some $k \in \mathbb{N}$ and $\infty \in U$ implies $\mathbb{N} \times \mathbb{N}\backslash([1,k] \times \mathbb{N} \cup (\mathbb{N} \times (\cup\{\mathbb{N}_i: 1 \leq i \leq k\}))) \subseteq U$ for some $k \in \mathbb{N}$. The space X is a noncompact, seminormal, HC space.

2. FFC and CFC Spaces

In this section two new concepts are introduced, characterized, and shown to lie strictly between the classes of minimal Hausdorff spaces and functionally compact spaces. The embedding problem is solved for FFC spaces and an example of FFC, but not CFC space, is shown to solve a problem by Vermeer [Ve].

A space X is called *finitely functionally compact* (resp. *compactly functionally compact*), abbreviated as FFC (resp. CFC), if every continuous function into a space with finite (resp. compact) point-inverses is closed.

(2.1) A FC space is CFC, a CFC space is FFC, and a FFC space is MH.

Proof. The proof follows from the facts that a singleton is a finite set, a finite set is compact, a compact set is θ-closed, and 1.3(c).

A subset A of a space X is called a *co*-H-*set* (resp. *co*-H-*closed*) if X\A is an H-set in X (resp. X\A is an HC subspace).

(2.2) *Proposition. Let* X *be a space. The following are equivalent:*

(a) *The space* X *is* MH *(resp.* FFC, CFC, FC*).*

(b) *Each singleton (resp. finite set, compact set, θ-closed set) has a local base consisting of co*-H-*sets.*

(c) *Each singleton (resp. finite set, compact set, θ-closed set) has a local base consisting of co*-H-*closed sets.*

(d) *The space* X *is* HC *and each singleton (resp. finite set, compact set, θ-closed set) has a local base consisting of regular open sets.*

Proof. Now, (c) implies (b) by 1.1(a) and (d) implies (c) as regular closed subsets of HC spaces are HC [K]. Also, (a) implies (d) follows immediately from 1.3(a), 1.2(c), and 2.1. If X is finite, then clearly (b) implies (a). So, suppose X is infinite and $p,q \in X$ such that $p \neq q$. There are disjoint open sets U and V such that $p \in U$ and $q \in V$. There are H-sets A,B in X such that $p \in X\backslash A \subseteq U$ and $q \in X\backslash B \subseteq V$. So, $X = A \cup B$ and by 1.1(c,d), X is an HC space. Let Y be a space and $f: X \to Y$ a continuous function such that for each $p \in Y$, $f^{-1}(p)$ is a singleton (resp. finite set, compact set, θ-closed set). Let $A \subseteq X$ be a closed set. Since X is H-closed, then by 1.1(a,f), $f[X]$ is closed in Y. To show $f[A]$ is closed in Y, it suffices to show $f[A]$ is closed in $f[X]$. So, let $p \in f[X]\backslash f[A]$. Then $f^{-1}(p) \cap A = \emptyset$. By (b), there is an H-set B such that $A \subseteq B$ and $f^{-1}(p) \cap B = \emptyset$. Since $f[B]$ is closed by 1.1(f), $p \notin f[B]$, and $f[A] \subseteq f[B]$, then it follows that $f[A]$ is closed.

The equivalence of 2.2(a) and (c) for MH spaces is noted in [FP].

It is well-known (see [PT]) that each space can be embedded in some MH space. We now answer the embedding question for an arbitrary space in FFC space; the analogous question will be answered for CC spaces in the next section. The embedding questions for CFC and FC remain unsolved.

(2.3) Let X be a space, $Y = \mathcal{S}(X)$, and $C \subseteq \mu Y$ such that $C\backslash Y$ is finite and $C \cap Y$ is compact, then C has a local base of co-H-closed sets. In particular, each space can be embedded in some FFC space.

Proof. Let U be an open set such that $C \subseteq U$. Since μY is H-closed, then by 1.4(c), it suffices to find a regular open set V in Y such that $C \subseteq oV \subseteq U$. Since $C \cap Y$ is compact, there is an open set T in Y such that $C \cap Y \subseteq T \subseteq U$ and $cl_{\mu Y}T \cap (C\backslash Y) = \emptyset$. Since $C \cap (X \times \{\infty\})$ is compact, there is some $m \in \mathbb{N}$ and open sets B_i in X and $n_i \in \mathbb{N}$ for $1 \leq i \leq m$ such that

(i) $B_i \times [n_i,\infty] \subseteq T$ for $1 \leq i \leq m$,

(ii) $o(B_i \times [n_i,\infty]) \subseteq U$ for $1 \leq i \leq m$, and

(iii) $C \cap (X \times \{\infty\}) \subseteq (\cup\{B_i: 1 \leq i \leq m\}) \times \{\infty\}$.

Let $B = \cup\{B_i: 1 \leq i \leq m\}$ and $k = \max\{n_i: 1 \leq i \leq m\}$. By 1.5(d) and 1.4 (g), $B \times [k,\infty]$ is regular open in Y and $o(B \times [k,\infty]) = \cup\{o(B_i \times [k,\infty]): 1 \leq i \leq m\}$. Hence, $C \cap (X \times \{\infty\}) \subseteq o(B \times [k,\infty]) \subseteq oT \cap U$.

Now, $A = (C \cap Y)\backslash(B \times [k,\infty])$ is compact and discrete; so, A is a finite clopen subset of Y. It follows that $A \cup (B \times [k,\infty])$ is a regular open subset of Y such that $C \cap Y \subseteq o(A \cup (B \times [k,\infty]) = A \cup o(B \times [k,\infty]) \subseteq oT \cap U$.

Let $C\backslash Y = \{\alpha_1,\cdots,\alpha_n\}$ for some $n \in \mathbb{N}$. Let $P_1,\cdots,P_{n+1}$ be a partition of $\mathbb{N}$ into infinite sets. For $i \leq n$, there is an unique $j(i) \leq n + 1$ such that $\alpha_i \in o(X \times P_{j(i)})$. Let $P = \cup\{P_{j(i)}: 1 \leq i \leq n\}$ and $Q = \mathbb{N}\backslash P$. Then P and Q are disjoint infinite subsets of $\mathbb{N}$. By 1.5(b), $X \times P$ is a regular open set in Y. Also, $C\backslash Y \subseteq o(X \times P)$. For each $i \leq n$, there is a regular open set U_i in Y such that $U_i \in \alpha_i$, $U_i \subseteq X \times P$, and $oU_i \subseteq (\mu Y\backslash cl_{\mu Y}T) \cap U$. Let $S = \cup\{U_i: 1 \leq i \leq n\}$. Since $S \subseteq X \times P$, then $int_Y cl_Y S \subseteq int_Y cl_Y(X \times P) \subseteq X \times P$. Since $cl_Y S\backslash S \subseteq cl_Y(X \times P)\backslash X \times P$, then S is regular open in Y. By 1.4(g), $oS = \cup\{oU_i: 1 \leq i \leq n\}$. So, $C\backslash Y \subseteq oS \subseteq (\mu Y\backslash cl_{\mu Y}T) \cap U$.

We now show that $V = A \cup (\dot{B} \times [k,\infty]) \cup S$ is regular open in Y. Since $cl_Y V \backslash V \subseteq X \times \{\infty\}$ by 1.5(c), let $(x,\infty) \in cl_Y V \backslash V$. Assume, by way of contradiction, that $(x,\infty) \in int_Y cl_Y V$. Then for some $t \in \mathbb{N}$, $\{x\} \times [t,\infty) \subseteq V$. Since $x \notin B$, then $(\{x\} \times [t,\infty)) \cap (B \times [k,\infty]) = \phi$. Thus, $(\{x\} \times [t,\infty))\backslash A \subseteq S$. Since $S \subseteq X \times P$, then $((\{x\} \times [t,\infty))\backslash A) \cap (\{x\} \times Q) = \phi$. This is a contradiction as Q is infinite. Hence, V is a regular open in Y, and by 1.4(g), we have that $oV = A \cup o(B \times [k,\infty]) \cup oS$. It follows that $C \subseteq oV \subseteq U$.

So, by 2.3, it appears that a positive solution for embedding each space in some CFC space is near. We do not know for what spaces X, when $\mu(\mathcal{S}(X))$ will be CFC. However, the next result shows that such spaces must be, at least, HC.

(2.4) Let X be a space. If $\mu(\mathcal{S}(X))$ is CFC, then X is HC.

Proof. Let $Y = \mathcal{S}(X)$, and let $\{U_a: a \in A\}$ be a regular open cover of X that does not contain a finite subfamily whose union is dense in X. Now, $U_a \times \mathbb{N}^*$ is a regular open subset of Y; so, $o(U_a \times \mathbb{N}^*)$ is regular open in μY by 1.4(c). Also, by 1.4(d) and 1.5(c), $cl_{\mu Y} o(U_a \times \mathbb{N}^*) = o(U_a \times \mathbb{N}^*) \cup ((cl_X U_a) \times \{\infty\})$. Let $W = \cup\{o(U_a \times \mathbb{N}^*): a \in A\}$. Now, $Y \subseteq W \subseteq \mu Y$ and $\mu Y \backslash W$ is closed in μY. Since Y is semi-regular, then by 1.4(f), $\mu Y \backslash W$ is compact. Assume there is a regular open set T in μY such $\mu Y \backslash W \subseteq T \subseteq \mu Y \backslash (X \times \{\infty\})$. Then $X \times \{\infty\} \subseteq \mu Y \backslash T \subseteq W$. But $\mu Y \backslash T$ is regular closed and, hence, is HC. So, there is a finite $F \subseteq A$ such that

$X \times \{\infty\} \subseteq \mu Y \backslash T \subseteq \cup\{cl_{\mu Y}(o(U_a \times \mathbb{N}^*)): a \in F\}$. This implies that $X \subseteq \cup\{cl_X U_a: a \in F\}$ which is a contradiction.

Now, it follows by 2.3 and 2.4 that there are FFC spaces which are not CFC, e.g., $\mu(\mathcal{S}(\mathbb{R}))$. Vermeer [Ve] has proven that if A is an H-set in a space X and $\{x_1,\cdots,x_n\} \subseteq X\backslash A$ for some $n \in \mathbb{N}$, then there is a regular open set V in X such that $\{x_1,\cdots,x_n\} \subseteq V \subseteq X\backslash A$ (this is a corollary of 1.2(a)). Vermeer asks whether this property can happen for all of the closed sets in a space which is not CC. The answer is positive in any FFC space which is not CFC (and, hence, not CC), e.g., in $\mu(\mathcal{S}(\mathbb{R}))$.

The space in 1.6(a) is a MH space but is not FFC. So, we now give an example of a CFC space which is not FC.

(2.5) *Example*. Let $X = (\mathbb{N} \times \mathbb{N} \times \mathbb{N}) \cup (\mathbb{N} \times \mathbb{N}) \cup \mathbb{N} \cup (\{\infty\} \times \mathbb{N}) \cup \{\pm\infty\}$. Let $\{P_i: i \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinite sets. Now, $U \subseteq X$ is defined to be open if

(i) $n \in U$ implies $(\{n\} \times \mathbb{N} \times \mathbb{N})\backslash F \subseteq U$ for some finite set $F \subseteq \{n\} \times \mathbb{N} \times \mathbb{N}$,

(ii) $\infty \in U$ implies $[k,\infty) \times [k,\infty) \times \mathbb{N} \subseteq U$ for some $k \in \mathbb{N}$,

(iii) $(\infty,n) \in U$ implies $([k,\infty) \times \{n\} \times \mathbb{N}) \cup (\{n\} \times [k,\infty)) \cup ([k,\infty) \times P_n) \subseteq U$ for some $k \in \mathbb{N}$, and

(iv) $-\infty \in U$ implies $\mathbb{N} \times \mathbb{N}\backslash(([1,k] \times \mathbb{N}) \cup (\mathbb{N} \times (P_1 \cup \cdots \cup P_k))) \subseteq U$ for some $k \in \mathbb{N}$.

We will show that X is a CFC space but is not FC. Note that the points of $(\mathbb{N} \times \mathbb{N} \times \mathbb{N}) \cup (\mathbb{N} \times \mathbb{N})$ are isolated in X. Let $Y = (\mathbb{N} \times \mathbb{N} \times \mathbb{N}) \cup \{\infty\} \cup \mathbb{N} \cup (\{\infty\} \times \mathbb{N})$ and $Z = (\mathbb{N} \times \mathbb{N}) \cup (\{\infty\} \times \mathbb{N}) \cup \{-\infty\}$.

First, we show that X is HC. Clearly, X is Hausdorff. It is easy to verify that Y is HC. The space Z is homeomorphic to the space in 1.6(b) and, hence, is CC. So, it follows that $X = Y \cup Z$ is HC by 1.1(a,c,d).

Next, we show that X is CFC. Let C be a compact subset of Y and A a closed subset of X such that $C \subseteq X \setminus Z$. By 2.2(d), we need to find a regular open set U in X such that $C \subseteq U \subseteq X \setminus A$. Since N and $\{\infty\} \times \mathbb{N}$ are closed sets, then $C \cap \mathbb{N}$ and $C \cap (\{\infty\} \times \mathbb{N})$ are compact and, hence, finite sets. Let $k \in \mathbb{N}$ be the smallest positive integer such that $C \cap (\mathbb{N} \cup (\{\infty\} \times \mathbb{N})) \subseteq [1,k] \cup (\{\infty\} \times [1,k])$. Do the following:

(a) if $\infty \in C$, pick $m_1 \in \mathbb{N}$ such that $m_1 > k$ and $V_1 \subseteq X \setminus A$ where $V_1 = \{\infty\} \cup ([m_1,\infty) \times [m_1,\infty) \times \mathbb{N})$,

(b) if $-\infty \in C$, pick $m_2 \in \mathbb{N}$ such that $m_2 > k$ and $V_2 \subseteq X \setminus A$ where $V_2 = \{-\infty\} \cup (\mathbb{N} \times \mathbb{N} \setminus [([1,m_2] \times \mathbb{N}) \cup (\mathbb{N} \times (P_1 \cup \cdots \cup P_{m_2})))])$,

(c) if $n \in C$, find a finite set $F_n \subseteq \{n\} \times \mathbb{N} \times \mathbb{N}$ such that $U_n \subseteq X \setminus A$ where $U_n = \{n\} \cup ((\{n\} \times \mathbb{N} \times \mathbb{N}) \setminus F_n)$, and

(d) if $(\infty,n) \in C$, pick $k_n \in \mathbb{N}$ such that $k_n > k$ and $W_n \subseteq X \setminus A$ where $W_n = \{(\infty,n)\} \cup ([k_n,\infty) \times \{n\} \times \mathbb{N}) \cup (\{n\} \times [k_n,\infty)) \cup ([k_n,\infty) \times P_n)$.

Since C is compact, there is a finite family $\mathcal{F} \subseteq \{V_1,V_2\} \cup \{U_n : 1 \le n \le k\} \cup \{W_n : 1 \le n \le k\}$ such that $F = C \setminus \cup\mathcal{F}$ is a finite subset of $(\mathbb{N} \times \mathbb{N} \times \mathbb{N}) \cup (\mathbb{N} \times \mathbb{N})$. So, $C \subseteq F \cup \cup\mathcal{F} \subseteq X \setminus A$. It is straightforward to verify that for any subset $\mathcal{G} \subseteq \{V_1,V_2\} \cup \{U_n : 1 \le n \le k\} \cup \{W_n : 1 \le n \le k\}$, $\cup\mathcal{G}$ is regular open in X. It follows that $F \cup \cup\mathcal{F}$ is also regular open in X.

Finally, we show that X is not FC. Let $A = \{\pm\infty\} \cup (\{\infty\} \times \mathbb{N})$. If $n \in \mathbb{N}$, then $(\{n\} \times \mathbb{N} \times \mathbb{N}) \cup \{n\}$ is a clopen set containing n and missing A. It follows that A is θ-closed in X. Now, $U = X\backslash\mathbb{N}$ is open and $A \subseteq U$. Let V be any open set such that $A \subseteq V \subseteq U$. Since $\infty \in V$, then for some $k \in \mathbb{N}$, $([k,\infty) \times [k,\infty) \times \mathbb{N}) \subseteq V$. For $1 \leq n \leq k$, there is some $k_n \in \mathbb{N}$ such that $([k_n,\infty) \times \{n\} \times \mathbb{N}) \subseteq V$. There is some $m \in \mathbb{N}$ such that $m \geq k$ and $m \geq k_n$ for $1 \leq n \leq k$. Then $\{m\} \times \mathbb{N} \times \mathbb{N} \subseteq V$ implying that $m \in (int_X cl_X V) \cap \mathbb{N}$. Since $V \subseteq X\backslash\mathbb{N}$, then V is not regular open in X.

Willard [W] has shown for each noncompact, FC space X, there is a compact space K such that X × K is not FC. This gives a negative answer to the product question about FC, CC, or seminormal, HC spaces. It is well-known (see [BPS]) that the product of MH spaces is MH. The product question for CFC spaces remain unsolved. However, the product question for FFC spaces is answered. We need the following characterization of FFC spaces.

(2.6) *Lemma. A space* X *is* FFC *iff* X *is* HC *and for any finite set* $F \subseteq U$ *where* U *is open in* X, *there is a pairwise disjoint family* $\{U_p\colon p \in F\}$ *of regular open sets in* X *such that* $p \in U_p \subseteq U$ *and* $\cup\{U_p\colon p \in F\}$ *is regular open.*

Proof. One direction is immediate by 2.2(d). To prove the converse, suppose X is FFC. By 2.2(d), X is HC. Let F be a finite subset of X and $F \subseteq U$ where U is an open subset of X. By Hausdorffness and semiregularity, we can find pairwise disjoint regular open sets $\{T_p\colon p \in F\}$

such that $p \in T_p \subseteq U$ for each $p \in F$. By 2.2(d), there is a regular open set V such that $F \subseteq V \subseteq \cup\{T_p: p \in F\}$. Let $U_p = V \cap T_p$ for $p \in F$. Then, for $p \in F$, U_p is regular open, $p \in U_p \subseteq U$, and $V = \cup\{U_p: p \in F\}$ is regular open.

(2.7) *Proposition. The product of* FFC *spaces is* FFC.

Proof. Let $\{X_a: a \in A\}$ be a family of FFC spaces and $X = \Pi\{X_a: a \in A\}$. Now, X is HC since the product of HC spaces is HC (see [BPS]). Let $F \subseteq U$ where F is finite and U is an open subset of X. For each $p \in F$, there is a finite subset $A_p \subseteq A$ and an open set U_p^a in X_a for each $a \in A_p$ such that $p \in V_p \subseteq U$ where $V_p = (\Pi\{U_p^a: a \in A_p\}) \times (\Pi\{X_b: b \in A\backslash A_p\})$. Let $B = \cup\{A_p: p \in F\}$ and $V = \cup\{V_p: p \in F\}$. Consider the projection map $\pi_B: X \to \Pi\{X_a: a \in B\}$. If there is a regular open set W in $\Pi\{X_a: a \in B\}$ such that $\pi_B[F] \subseteq W \subseteq \pi_B[V]$, then $F \subseteq \pi_B^{-1}[W] \subseteq V$ and $\pi_B^{-1}[W] = W \times \Pi\{X_a: a \in A\backslash B\}$ is regular open in X. Thus, it suffices to show that a finite product of FFC spaces is FFC; however, by induction, it reduces to show that the product of two FFC spaces is FFC.

Let X and Y be FFC spaces, $Z = X \times Y$, and $F \subseteq U \subseteq Z$ where F is finite and U is an open subset in Z. Since $\pi_X[F] \subseteq \pi_X[U]$ (resp. $\pi_Y[F] \subseteq \pi_Y[U]$), there are regular open sets $\{T_p: p \in F\}$ (resp. $\{S_p: p \in F\}$) in X (resp. in Y) such that $\cup\{T_p: p \in F\}$ (resp. $\cup\{S_p: p \in F\}$) is regular open in X (resp. Y), $\pi_X(p) \in T_p \subseteq \pi_X[U]$ (resp. $\pi_Y(p) \in S_p \subseteq \pi_Y[U]$), and if for $p, q \in F$, $\pi_X(p) \neq \pi_X(q)$ (resp. $\pi_Y(p) \neq \pi_Y(q)$) implies $T_p \cap T_q = \emptyset$ (resp. $S_p \cap S_q = \emptyset$) and $\pi_X(p) = \pi_X(q)$ (resp. $\pi_Y(p) = \pi_Y(q)$) implies $T_p = T_q$ (resp. $S_p = S_q$).

Also, for each $p \in F$, $p \in T_p \times S_p \subseteq U$. Let $W = \{T_p \times S_p : p \in F\}$. The proof is completed when we show that W is regular open. Let $(x,y) \in int_Z cl_Z W$. These are open sets A in X and B in Y such that $(x,y) \in A \times B \subseteq \cup\{(cl_X T_p) \times (cl_Y S_p): p \in F\}$. So, $x \in A \subseteq \cup\{cl_X T_p: p \in F\}$ implying that $x \in int_X cl_X(\cup\{T_p: p \in F\}) = \cup\{T_p: p \in F\}$ and that $x \in T_p$ for some $p \in F$. Likewise, $y \in S_q$ for some $q \in F$. If $\pi_X(p) = \pi_X(q)$, then $(x,y) \in T_p \times S_q = T_q \times S_q \subseteq W$; likewise, if $\pi_Y(p) = \pi_Y(q)$, then $(x,y) \in T_p \times S_p \subseteq W$. If $\pi_X(p) \neq \pi_X(q)$ and $\pi_Y(p) \neq \pi_Y(q)$, then it follows that $(T_p \times S_q) \cap W = (T_p \times S_q) \cap (\cup\{T_r \times S_r: r \in F\}) = \emptyset$ contradicting that $(x,y) \in cl_Z W$. Thus, W is regular open in Z.

A space X is called *rim-H-closed* if there is an open base with H-closed boundaries.

(2.8) A rim-H-closed, HC space X is FFC.

Proof. Let $x_1, \cdots, x_n$ be points in X and W be an open subset of X such that $\{x_1, \cdots, x_n\} \subseteq W$. We can find pairwise disjoint open sets $U_1, \cdots, U_n$ such that $x_i \in U_i \subseteq W$ and bdU_i is HC for $i = 1, \cdots, n$. Now, $H = \cup\{bdU_i: 1 \leq i \leq n\}$ is HC. By 1.2(a), there is a regular open set V in X such that $\{x_1, \cdots, x_n\} \subseteq V \subseteq X \backslash H$. Let $U = \cup\{U_i: 1 \leq i \leq n\}$. Note that $\{x_1, \cdots, x_n\} \subseteq U \cap V \subseteq W$. Now, $int_X cl_X(U \cap V) = int_X cl_X(U) \cap int_X cl_X(V) = int_X cl_X(U) \cap V$. But $int_X cl_X U \subseteq (bdU) \cup U$ and $V \cap bdU = \emptyset$. Thus, $int_X cl_X U \cap V = U \cap V$ implying that $U \cap V$ is regular open.

(2.9) *Remarks.*

(a) One can prove 2.8 for an HC space with an open base whose boundaries are H-sets.

(b) Since FFC spaces are MH, then by 2.8 it follows that rim-H-closed, HC spaces are MH.

(c) The space in 1.6(b) is an example of a seminormal, HC (and, hence FFC) space which is not rim-H-closed.

(d) Pettey [P] has proven that every HC space X is the perfect, continuous, open retraction of a rim-H-closed, HC space Y; in particular, X is a subspace of Y.

(e) Let X be a space and Z be an HC space containing X. By (d) and 2.8, Z is a subspace of an FFC space. Even though this proof that every space can be embedded in some FFC space is shorter than the proof of 2.3, 2.3 is presented because actually more is obtained (CFC except for the remainder), 2.3 and 2.4 produce many examples of FFC spaces which are not CFC, and the space generated in the embedding problem of the third section uses the construction of 2.3.

(f) Herrlich (see [BPS]) has given an example of a MH space which is not of second category; in fact, if X is the MH space, there is a family $\{U_n: n \in \mathbb{N}\}$ of open, dense sets such that $\cap\{U_n: n \in \mathbb{N}\} = \emptyset$. By (d) and 2.8, X is the continuous, open image of a FFC space Y. Let $f: Y \to X$ be the continuous, open surjection, and let $D_n = f^{-1}[U_n]$ for $n \in \mathbb{N}$. Now, D_n is open by continuity and dense since f is open. Since $f[\cap\{D_n: n \in \mathbb{N}\}] \subseteq \cap\{U_n: n \in \mathbb{N}\} = \emptyset$, then $\cap\{D_n: n \in \mathbb{N}\} = \emptyset$. Hence, Y is an example of a FFC space which is not of second category. Lim and Tan [LT] have shown that a seminormal, HC space is of second category. It is not known whether CFC, FC, or CC spaces are of second category.

3. FC, CC, and Seminormal, HC Spaces

In this section, we give a new characterization of CC spaces and a "closed function" characterization which extends to CC spaces. A question by Lim and Tan [LT] about FC spaces is answered. An example of a space which can not be embedded in any CC space is given.

The characterizations of FC spaces relative to θ-closed sets in 2.2(b), (c), and (d) motivate us to investigate the corresponding analogues for CC spaces relative to closed sets. Since a space is CC if each open set is a co-H-set, then, at first glance, it appears that the class of spaces satisfying 2.2(b) relative to closed sets properly contains the class of CC spaces. Surprisingly, as shown in the next result, the classes coincide.

(3.1) A space X is CC iff every closed set of X has a base consisting of co-H-sets.

Proof. The proof in one direction is obvious. Conversely, suppose every closed subset of X has a base consisting of co-H-sets. Let A be a closed set, and let $\mathcal{F}$ be an open filter base on X such that $\mathcal{F}$ meets A, i.e., $F \cap A \neq \emptyset$ for each $F \in \mathcal{F}$. Assume, by way of contradiction, that $A \cap ad_X\mathcal{F} = \emptyset$ (see 1.1(e)). Then there is a co-H-set V such that $ad_X\mathcal{F} \subseteq V \subseteq X\backslash A$. Since $X\backslash V \supseteq A$, then $\mathcal{F}$ meets $X\backslash V$. Since $X\backslash V$ is an H-set, then by 1.1(e), $ad_X\mathcal{F} \cap X\backslash V \neq \emptyset$. This is a contradiction as $ad_X\mathcal{F} \subseteq V$.

The condition 2.2(d) relative to closed sets, i.e., X is HC and every closed set has a local base of regular open sets is precisely the definition of seminormal and HC. Since there are CC spaces (see [GV_2]) which are not seminormal, then 2.2(b) relative to closed sets is not equivalent to 2.2(d) relative to closed sets. 3.1 does not extend to seminormal, HC spaces as it is well-known [K] that a space in which every open set is co-H-closed is compact. What remains is examining spaces for which 2.2(c) relative to closed sets is satisfied. Consider this condition:

(*) every closed set has a local base of co-H-closed sets.

Clearly, a seminormal HC space has property (*), and a space satisfying property (*) is CC. The example in [GV_2] of a CC space which is not seminormal has property (*).

The characterization of MH, FFC, CFC, and FC spaces in terms of certain continuous functions being closed does not extend to CC spaces, for, by 1.2(d), if $f: X \to Y$ is continuous, where X and Y are spaces, then $f^{-1}(p)$ is θ-closed in X for each $p \in Y$. So, to characterize CC spaces in terms of functions, we need to either drop the condition that the range space is Hausdorff (done by Willard [W]) or reduce the continuity condition. We can accomplish the latter by defining a function $f: X \to Y$, where X and Y are spaces, to be *barely continuous* if for each H-set A in X, f[A] is closed. Note that by 1.1(f), θ-continuous functions are barely continuous.

(3.2) A space X is MH (resp. FFC, CFC, FC, CC) iff every barely continuous function from X into a space in which point-inverses are singletons (resp. finite, compact, θ-closed, closed) is closed.

Proof. The proofs for MH, FFC, CFC, and FC spaces are similar. The proof in one direction is essentially the same proof as the second half of the proof of "(b) implies (a)" in 2.2. The proof in the other direction is easy as continuous functions are barely continuous.

Suppose X is CC. Since every closed set is an H-set, then every barely continuous function with X as domain is closed. Conversely, suppose every barely continuous function from a space X into a space in which point-inverses are closed is closed. Suppose $A \subseteq U \subseteq X$ where A is closed and U is open in X. Consider the set X/A and the function $p_A: X \to X/A$ defined after 1.1. Instead of using the quotient topology on X/A (which may not be Hausdorff by 1.2(b)), define a topology on X/A by $V \subseteq X/A$ is open if $\infty \in V$ implies $(X/A)\backslash V$ is an H-set in X (note that $(X/A)\backslash V$ is a subset of X when $\infty \in V$). In particular, each point of $(X/A)\backslash\{\infty\}$ is clopen. Thus, X/A is Hausdorff. Also, p_A is barely continuous, and point-inverses of p_A are closed in X. Since p_A is closed, then $p_A[X\backslash U]$ is closed in X/A. Since $\infty \in (X/A)\backslash p_A[X\backslash U]$, then $p_A[X\backslash U] = X\backslash U$ is an H-set in X. By 3.1, X is a CC space.

Lim and Tan [LT] have defined a subset A of a space X to have property C(1) in X if for every continuous function f from X into a space, f[A] is closed. They asked this question:

> If A has property C(1) in a space X and $\mathcal{C}$ is a cover of A by sets regular open in X, is there a finite subfamily $\mathcal{F} \subseteq \mathcal{C}$ such that $A \subseteq \cup\{cl_X C: C \in \mathcal{F}\}$?

To answer this question we will use the next result.

(3.3) Let X be a space and $A \subseteq X$. Then A is an H-set in X iff for each cover $\mathcal{C}$ of A by regular open subsets of X, there is a finite subfamily $\mathcal{F} \subseteq \mathcal{C}$ such that $A \subseteq \cup\{cl_X U: U \in \mathcal{F}\}$.

Proof. The proof in one direction is immediate from the definition of an H-set. Conversely, suppose $\mathcal{C}$ is a cover of A by open subsets of X. Then $\{int_X cl_X U: U \in \mathcal{C}\}$ is a cover of A by regular open subsets of X. So, there is a finite subset $\mathcal{F} \subseteq \mathcal{C}$ such that $A \subseteq \cup\{cl_X(int_X cl_X U): U \in \mathcal{F}\}$. But $cl_X(int_X cl_X U) = cl_X U$. Hence, $A \subseteq \cup\{cl_X U: U \in \mathcal{F}\}$, and A is an H-set.

In view of 3.3, the question in [LT] becomes, "If a subset A has property C(1) in a space X, is it true that A is an H-set in X?" Let X be a FC space which is not CC (see [GV_1, Ex. 4; LT, Ex. 3]). Then there is a closed subset A in X which is not an H-set. However, since X is a FC space, then A has property C(1) in X. Consequently, the answer is no.

To decide whether each space is embeddable in some CC space, we present a result that slightly extends a result by Vermeer [Ve].

(3.4) If $\mathbb{Q}$ is a subspace of a space X and $\mathbb{Q} \subseteq A \subseteq X$ where A is an H-set in X, then A is not countable.

Proof. Suppose A is countable. Let $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ and $A\backslash\mathbb{Q} = \{p_n : n \in \mathbb{N}\}$. Let U_1 be an open subset of X such that $q_2 \in U_1$ and $\{p_1, q_1\} \cap cl_X U_1 = \phi$. Find an open subset U_2 of U_1 such that $\{p_2, q_2\} \cap cl_X U_2 = \phi$ and $U_2 \cap \mathbb{Q} \neq \phi$. Continue by induction to construct a chain $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ of open subsets of X such that $U_n \cap \mathbb{Q} \neq \phi$ and $\{p_n, q_n\} \cap cl_X U_n = \phi$ for each $n \in \mathbb{N}$. Now, $\mathcal{F} = \{U_n : n \in \mathbb{N}\}$ is an open filter base on X such that $U_n \cap A \neq \phi$ for each $n \in \mathbb{N}$ and $A \cap ad_X \mathcal{F} = \phi$. By 1.1(e), A is not an H-set in X.

An immediate corollary of 3.4 is the result by Vermeer [Ve] that Q is not an H-set of any space.

(3.5) *Examples*

(a) There is an HC space which is not a subspace of any CC space. Let $X = \mu(\mathcal{S}(\mathbb{Q}))$. By 1.5(b), $\mathbb{Q}$ is a closed subset of X. Suppose X is a subspace of some space Z. Then, since X is HC, X is a closed subset of Z. So, $\mathbb{Q}$ is a closed subspace of Z. By 3.4, $\mathbb{Q}$ is not an H-set of Z. Hence, Z is not a CC space.

(b) There is a seminormal space which is not a subspace of any CC space. Let $X = \mu(\mathcal{S}(\mathbb{Q}))$ and $Y = \mathcal{S}(X)$. Then Y is seminormal by 1.5(a). If Y is a subspace of some space Z, then X is a subspace of Z and by (a), Z is not a CC space.

4. Unsolved Problems

The first group of problems are considered "embedding problems." By 2.3, each space can be embedded in some FFC space. It is known (see [BPS]) that each space can be densely embedded in some HC space and a space is densely embeddable in some MH space iff it is semiregular. It is easy to verify that a necessary condition for a space to be densely embedded in an FFC (resp. CFC) space is for every finite (resp. compact) set to have a base of regular open sets.

(1) Find a necessary and sufficient condition for a space to be densely embeddable in some FFC space.

On the other hand, we know that there are spaces which can not be embeddable in any CC spaces and, hence, not in any seminormal, HC spaces.

(2) Find a necessary and sufficient condition for a space to be embeddable in some CC space or in some seminormal, HC space.

The basic embedding problem for FC and CFC spaces remains unsolved.

(3)(a). Can each space be embedded in some CFC space?

(b) If the answer to (a) is yes, then can each space be embedded in some FC space?

We suspect the answer to 3(a) is negative. In fact, if $\mathbb{Q}$ is the space of rational numbers, $U_n = \{r \in \mathbb{Q}: n\pi < r < (n+1)\pi\}$ for $n \in \mathbb{Z}$, and $Y = \mathcal{S}(\mathbb{Q})$, then $C = \mu Y \setminus \cup\{o(U_n \times \mathbb{N}^*): n \in \mathbb{Z}\}$ is compact by 1.4(f). If μY

is a subspace of a space Z, then $C \subseteq Z \backslash (\mathbb{Q} \times \{\infty\})$ and $Z \backslash (\mathbb{Q} \times \{\infty\})$ is open in Z. We suspect there is no regular open set V in Z such that $C \subseteq V \subseteq Z \backslash (\mathbb{Q} \times \{\infty\})$; if so, then μY is not embeddable in any CFC space.

The major unsolved problem in products is in the class of CFC spaces.

(4) Is the product of CFC spaces a CFC space?

By 2.9(f), there are FFC spaces which are not of second category, and by a result in [LT], every seminormal, HC space is of second category. We do not know about the "second category" status of CFC, FC and CC spaces.

(5) Is every CFC space, FC space, or CC space of second category?

By 2.8 and 2.9(d), every HC space is the open, perfect, continuous retraction of a FFC space. It is easy to show that the continuous image of a FC space is FC and the perfect continuous image of a CFC space is CFC.

(6)(a) Is each FFC space the continous image of some CFC space?

(b) If the answer to (a) is yes, then is each FFC space the continuous open image or retraction of a CFC space?

If each FFC space is the open continuous image of a CFC space, then using the technique in 2.9(f), we obtain a CFC space which is not of second category. If each FFC space is the retraction of a CFC space, then each space is embeddable in a CFC space.

References

[BPS] M. P. Berri, J. R. Porter and R. M. Stephenson, Jr., *A survey of minimal topological spaces*, Proc. Kanpur Topology Conf., 1968, Academic Press (1970), 93-114.

[DP] R. F. Dickman and J. R. Porter, *θ-closed subsets of Hausdorff spaces*, Pac. J. Math. 59 (1975), 407-415.

[DZ] R. F. Dickman and A. Zame, *Functionally compact spaces*, Pac. J. Math. 31 (1969), 303-311.

[FP] L. M. Friedler and D. H. Pettey, *Inverse limits and mappings of minimal topological spaces*, Pac. J. Math. 71 (1977), 429-448.

[GV_1] G. Goss and G. Viglino, *Some topological properties weaker than compactness*, Pac. J. Math. 35 (1970), 635-638.

[GV_2] ________, C-*compact and functionally compact spaces*, Pac. J. Math. 37 (1971), 677-681.

[K] M. Katětov, *Über* H-*abgeschlossene and bikompakt Räume*, Časopis pěst. mat. 69 (1940), 36-49.

[LT] T.-C. Lim and K.-K. Tan, *Functional compactness and* C-*compactness*, J. London Math. Soc. (2) 9 (1974), 371-377.

[P] D. H. Pettey, *Locally* P-*closed spaces and rim* P-*closed spaces*, Proc. Amer. Math. Soc. 87 (1983), 543-548.

[Po] J. R. Porter, *Lattices of* H-*closed extensions*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 22 (1974), 831-837.

[PT] ________ and J. D. Thomas, *On* H-*closed and minimal Hausdorff spaces*, Trans. Amer. Math. Soc. 138 (1969), 159-170.

[PV_1] J. R. Porter and C. Votaw, H-*closed extensions* I, Gen. Top. and Appl. 3 (1973), 211-224.

[PV_2] ________, H-*closed extensions* II, Trans. Amer. Math. Soc. 202 (1975), 193-209.

[S] S. Sakai, *A note on* C-*compact spaces*, Proc. Japan Acad. 46 (1970), 917-920.

[Ve] J. Vermeer, *Closed subspaces of* H-*closed spaces*, Pac. J. Math. 118 (1985), 229-247.

[V_1] G. Viglino, C-*compact spaces*, Duke Math. 36 (1969), 761-764.

[V_2] ________, *Seminormal and* C-*compact spaces*, Duke Math. 38 (1971), 57-61.

[W] S. W. Willard, *Functionally compact spaces*, C-*compact spaces, and mappings of minimal Hausdorff spaces*, Pac. J. Math. 38 (1971), 267-272.

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