TOPOLOGY PROCEEDINGS
Volume 9, 1984
Pages 269–291

http://topology.auburn.edu/tp/

ON ULTRA POWERS OF BOOLEAN ALGEBRAS

by

Alan Dow

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

ON ULTRA POWERS OF BOOLEAN ALGEBRAS[1]

Alan Dow

0. Introduction

If A is an algebra with finitely many finitary operations and relations and if p is an ultrafilter on ω then the reduced ultrapower A^{ω}/p is also an algebra with the same operations. Keisler has shown that CH implies A^{ω}/p is isomorphic to A^{ω}/q for any free ultrafilters p,q on ω when $|A| \leq \underline{c}$. In this note it is shown that if CH is false then there are two free ultrafilters p,q on ω such that if $(A,<)$ has arbitrarily long finite chains then A^{ω}/p is not isomorphic to A^{ω}/q. This answers a question in [ACCH] about real-closed η_1-fields. Furthermore we show that, if A is an atomless boolean algebra of cardinality at most $\underline{c}$, then each ultrafilter of A^{ω}/p has a disjoint refinement, partially answering a question in [BV]. We also show that if B is the countable free boolean algebra then it is consistent that there is an ultrafilter p on ω so that $P(\omega)/\text{fin}$ will embed into B^{ω}/p but B^{ω}/p will not embed into $P(\omega)/\text{fin}$.

1. Preliminaries

In this section the notation we use is introduced and we review some facts about ultraproducts which we will require. Our standard reference is the Comfort and Negrepontis text [CN]. Small Greek letters will denote ordinals

[1]Research supported by the NSERC of Canada.

and a cardinal is an initial ordinal. If S is a set and α is an ordinal, then S^α is the set of functions from α to S, $|S|$ is the cardinality of S and $[S]^{<\alpha}$ is the set of subsets of S of cardinality less than α. We sometimes use 2^α to denote cardinal exponentiation and this shall be clear from the context. If an ultrafilter p on a cardinal α has the property that $|A| = \alpha$ for each $A \in p$ then p is called a *uniform* ultrafilter; $U(\alpha)$ is the set of all uniform ultrafilters on α, $\beta\alpha$ is the set of all ultrafilters on α and α^* is all free ultrafilters.

Let α be an infinite cardinal and let $p \in \alpha^*$, for a set S the ultrapower S^α/p is the set of equivalence classes on S^α where for $s,t \in S^\alpha$, $s =^p t$ if $\{a \in \alpha: s(a) = t(a)\} \in p$. We will usually assume that when we choose $s \in S^\alpha/p$ we have in fact chosen $s \in S^\alpha$. If $L(,)$ is a binary relation on S then $L(p, ,)$ is a relation on S^α/p or S^α defined by $L(p,s,t)$ if $\{a \in \alpha: L(s(a),t(a))\} \in p$. More generally, if p is any filter on α, define $L(p,s,t)$ if $\{a \in \alpha: L(s(a), t(a))\} \in p$. If for $\gamma \in \alpha$, S_γ is a set then the ultraproduct $\Pi_{\gamma<\alpha} S_\gamma/p$ is defined similarly, as are any relations and functions. Also let $L(p,s,t,v)$ abbreviate $L(p,s,t)$ and $L(p,t,v)$. Throughout this paper L will be an order (the usual order on an ordinal) and E will be equality.

A function V from $[\alpha]^{<\omega}$ to $P(\alpha)$ is called *multiplicative* if $V(H) = \{V(\{\alpha\}): a \in H\}$ for each $H \in [\alpha]^{<\omega}$. A filter p on α is called α^+-*good* if for each function W from $[\alpha]^{<\omega}$ to p there is a multiplicative function V from $[\alpha]^{<\omega}$ to p such that $V(H) \subseteq W(H)$ for each $H \in [\alpha]^{<\omega}$. A filter is ω-*incomplete* if it has countable many members whose intersection is empty.

A structure (S,L) is α-*saturated* if whenever fewer than α sentences of the form $\exists x\ L(s,x)$, $\exists x \neg L(s,x)$, $\exists x\ L(x,s)$ or $\exists x \neg L(x,s)$ are given and any finitely many can be satisfied with a single $x \in S$, then there is an $x \in S$ which satisfies them all simultaneously. For example the set of rationals with the usual order is ω-saturated but not ω_1-saturated. For subsets C,D of S, let $L(C,D)$ abbreviate that $L(c,d)$ for each $c \in C$ and $d \in D$, in case of $L(C,\{d\})$ or $L(\{c\},D)$ we will omit the parentheses. For regular cardinals κ,λ we say that (C,D) forms a (κ,λ)-*gap* in (S,L) if $L(C,D)$, C is an increasing chain of order type κ, D is a decreasing chain with order type λ under the reverse ordering and there is no $x \in S$ with $L(C,x,D)$.

Keisler introduced the notion of an α^+-good ultrafilter basically because of the following theorem. Keisler showed that assuming GCH there are ω-incomplete α^+-good ultrafilters in $U(\alpha)$ and Kunen later removed the GCH assumption (see [Ke], [K], [CN]).

1.1 *Theorem* (Keisler). $(S^\alpha/p, L(p))$ *is* α^+-*saturated if* (S,L) *is* ω-*saturated and* $p \in U(\alpha)$ *is* ω-*incomplete and* α^+-*good.*

Another result of Keisler's which we require is the following.

1.2 *Theorem* (Keisler). *If* $p \in U(\alpha)$ *is* α^+*-good and* $\{S_\gamma : \gamma < \alpha\}$ *are all finite sets such that* $\{\{\gamma : |S_\gamma| > n\} : n \in \omega\} \subset p$ *then* $|\Pi_{\gamma<\alpha} S_\gamma / p| = 2^\alpha$. *(Note that* p *is* ω*-incomplete.)*

We include a proof of 1.2 because it is probably not as well known as 1.1 and to give the flavor of the use of good filters.

Proof. Let W be the map from $[\alpha]^{<\omega}$ to p defined by $W(H) = \{\gamma : |S_\gamma| > k\}$ where $k = |H^H|$. Suppose that $V : [\alpha]^{<\omega} \to p$ is a multiplicative function refining W. For each $\gamma < \alpha$, let $H_\gamma = \{\delta \in \alpha : \gamma \in V(\{\delta\})\}$. Now define $n_\gamma = |H_\gamma|$ and note that we may assume that $S_\gamma \supset T_\gamma = n_\gamma^{H_\gamma}$ since $V(H_\gamma) = \{V(\{\delta\}) : \delta \in H_\gamma\} \subset W(H_\gamma)$. Let $X = \Pi_{\gamma<\alpha} n_\gamma / p$. Define a function e from X^α to $\Pi_{\gamma<\alpha} T_\gamma / p$ as follows: for $y \in X^\alpha$ let $e(y) \in \Pi_{\gamma<\alpha} T_\gamma / p$ where $e(y)(\gamma) \in T_\gamma$ and is such that $e(y)(\gamma)(\delta) = y(\delta)(\gamma)$ for each $\delta \in H_\gamma$. Now if $y \neq z$ are both in X^α, then for some $\delta \in \alpha\ \neg E(p, y(\delta), z(\delta))$. It follows that $\{\gamma \in \alpha : e(y)(\gamma) \neq e(z)(\gamma)\} \supset \{\gamma \in \alpha : \delta \in H_\gamma$ and $y(\delta) \neq z(\delta)\} = V(\{\delta\}) \cap \{\gamma : y(\delta)(\gamma) \neq z(\delta)(\gamma)\} \in p$ and so $e(y) \neq e(z)$. Therefore $|\Pi_{\gamma<\alpha} S_\gamma / p| \geq |\Pi_{\gamma<\alpha} T_\gamma / p| \geq |X^\alpha| = 2^\alpha$. The reverse inequality is trivial.

1.3 *Definition*. For a cardinal α, let $\underline{\gamma} \in \alpha^\alpha$ where $\underline{\gamma}(\delta) = \gamma$ for $\delta \in \alpha$. For $p \in U(\alpha)$, define $\kappa(i,p) = \min\{\kappa : (\alpha^\alpha, L(p))$ has an (ω_i, κ)-gap of the form $(\{\underline{\gamma} : \gamma < \omega_i\}, \{f_\delta : \delta < \kappa\})\}$ for each regular $\omega_i \leq \alpha$. Similarly, let $b(p) = \min\{\kappa : (\alpha^\alpha, L(p))$ has $(\kappa, \emptyset)$-gap$\}$. If $\alpha = \omega$, let $\kappa(\emptyset, p) = \kappa(p)$.

1.4 *Proposition. Let* $p \in U(\alpha)$ *be* ω*-incomplete* α^+*-good. If* (S,L) *has increasing chains of any finite length then, for each regular* $\omega_i \leq \alpha$, $\kappa(i,p)$ *is the unique regular cardinal such that* $(S^\alpha, L(p))$ *has an* (ω_i,κ)*-gap. Hence* $\kappa(i,p) > \alpha$.

Proof. Let us first show that $(S^\alpha, L(p))$ has an increasing chain of order type α. Fix $\{A_n: n \in \omega\} \subset p$ so that $\cap A_n = \emptyset$ and let V be a multiplicative map of $[\alpha]^{<\omega}$ into p with $V(H) \subset A_{|H|}$ for $H \in [\alpha]^{<\omega}$. For each $\delta \in \alpha$, let $H_\delta = \{\gamma \in \alpha: \delta \in V(\{\gamma\})\}$ and let $C_\delta = \{c(\delta,\gamma): \gamma \in H_\delta\} \subset S$ be a chain. Define, for $\gamma \in \alpha$, $g_\gamma \in S^\alpha$ so that if $\gamma \in H_\delta$ then $g_\gamma(\delta) = c(\delta,\gamma)$. Now if $\beta < \gamma < \alpha$, then $\{\delta \in \alpha: L(g_\beta(\delta), g_\gamma(\delta))\} \supset V(\{\beta,\gamma\}) \in p$. It is now clear that if $\omega_i \leq \alpha$ is regular and $\{g_\gamma: \gamma < \omega_i\} \subset S^\alpha$ is a chain then we may assume that V, $\{g_\gamma: \gamma < \omega_i\}$, $\{H_\delta: \delta \in \alpha\}$ and $\{C_\delta: \delta \in \alpha\}$ are as above. Furthermore if $h \in S^\alpha$ is such that $L(p,g_\gamma,h)$ for $\gamma < \omega_i$ then there is an $h' \in \Pi_{\delta<\alpha} C_\delta$ so that $L(p,g,h',h)$ for $\gamma < \omega_i$. Indeed, define $h'(\delta) = \max\{g_\gamma(\delta): \gamma \in H_\delta$ and $L(g_\gamma(\delta),h(\delta))\}$. Therefore, for any regular cardinal κ, if $\{h_\gamma: \gamma < \kappa\} \subset S^\alpha$ is such that $(\{g_\gamma: \gamma < \omega_i\}, \{h_\gamma: \gamma < \kappa\})$ form a gap, then we may assume $\{h_\gamma: \gamma < \kappa\} \subset \Pi_{\delta<\alpha} C_\delta$. Similarly in the structure $(\alpha^\alpha, L(p))$, if $\{f_\gamma: \gamma < \kappa\} \subset \alpha^\alpha$ is such that $(\{\underline{\gamma}: \gamma < \omega_i\}, \{f_\gamma: \gamma < \kappa\})$ form a gap, we may assume $f_\gamma \in \Pi_{\delta<\alpha} H_\delta$. The result now follows from the fact that $(\Pi_{\delta<\alpha} C_\delta/p, L(p))$ is isomorphic to $(\Pi_{\delta<\alpha} H_\delta/p, L(p))$.

If B is a boolean algebra, then the *Stone space* of B, $S(B)$, is the space of ultrafilters of B in which a set is closed and open (=clopen) precisely when it is of the form

$b^* = \{p \in S(B): b \in p\}$. Conversely if X is a compact space with a base for the topology consisting of clopen sets (= 0-dimensional) then $CO(X)$ is the boolean algebra of clopen subsets of X. It is clear that B is isomorphic to $CO(S(B))$ and that X is homeomorphic to $S(CO(X))$. Also B embeds into $CO(X)$ if and only if X maps continuously onto $S(B)$. The set $\beta\alpha$ is topologized as $S(P(\alpha))$ and both $U(\alpha)$ and α^* have the subspace topology. Recall that the unique countable atomless boolean algebra is equal to $CO(2^\omega)$ where 2^ω is the Cantor set (i.e. 2^ω has the product topology).

There is an alternate construction of an ultrapower of a boolean algebra B. The topological space $\alpha \times S(B)$ (where α has the discrete topology) has a Stone-Cech compactification $\beta(\alpha \times S(B))$. In fact, $\beta(\alpha \times S(B))$ is just the Stone space of B^α. The map $f\colon \alpha \times S(B) \to \alpha$ defined by $f[\{\gamma\} \times S(B)] = \{\gamma\}$ extends to an open map f from $\beta(\alpha \times S(B))$ to $\beta\alpha$. If we let $K^p = f^{\leftarrow}(p)$ for $p \in U(\alpha)$ then $CO(K^p) \cong B^\alpha/p$. If p is ω-incomplete α^+-good then K^p is an F_{α^+}-space in which any non-empty intersection of at most α many clopen sets has infinite interior (see [CN]). This is clearly not a useful way of constructing the ultrapower but the space K^p is an interesting topological space and an analogous construction can be made from spaces of the form $\alpha \times Y$ where Y is, for example, connected.

2. The Main Constructions

Let (S,L) be an ω-saturated structure with $|S| = \alpha$ and let $p,q \in U(\alpha)$ be ω-incomplete α^+-good. If $2^\alpha = \alpha^+$ then it is easily seen by 1.1 that $S^\alpha/p \cong S^\alpha/q$ because they each have cardinality 2^α. However if $2^\alpha > \alpha^+$ it may not be the case that these ultrapowers are isomorphic. The easiest way to distinguish them would be if $\kappa(i,p) \neq \kappa(i,q)$ for some $\omega_i \leq \alpha$. In this section we show that there is always $p \in U(\alpha)$ so that $\kappa(i,p) = cf(2^\alpha)$ (the cofinality of 2^α) for each $\omega_i \leq \alpha$. Furthermore in the case of $\alpha = \omega$ we show that $\kappa(p)$ can be anything reasonable. In fact we prove the following two theorems.

2.1 *Theorem. There is an* ω*-incomplete* α^+*-good ultrafilter* p *on* α *so that* $\kappa(i,p) = cf(2^\alpha)$ *for each regular* $\omega_i \leq \alpha$.

2.2 *Theorem. For each regular* κ *with* $\omega_1 \leq \kappa \leq 2^\omega$ *there is a* $p \in U(\omega)$ *so that* $\kappa(p) = \kappa$.

The reason that we are able to prove more for $\alpha = \omega$ is that every free ultrafilter on ω is ω^+-good which is not the case for $\alpha > \omega$. If $(R,<,+,x)$ is the field of real numbers then $(R^\omega/p, L(p), +(p), x(p))$ with the obvious meanings is an example of a real-closed η_1-field or an H-field (see [ACCH]), for each p in $U(\omega)$. From 2.2, we obtain the following answer to a question in [ACCH].

2.3 *Corollary. If* $2^\omega > \omega_1$ *then there are non-isomorphic H-fields of cardinality* 2^ω. *These fields may all have the form* R^ω/p *for* $p \in U(\omega)$.

This was shown to be consistent by Roitman [R] and the first sentence was shown to be consistent in [ACCH].

Recall that if p is a filter on α, not necessarily maximal, and $f,g \in S^\alpha$ then $L(p,f,g)$ denotes the condition $\{\delta < \alpha: L(f(\delta),g(\delta))\} \in p$. If p is the cofinite filter on ω then we use $f <^* g$ rather than $L(p,f,g)$. Recall that $\underline{b} = \min\{|F|: F \subset \omega^\omega$ and there is no $g \in \omega^\omega$ such that $f <^* g$ for all $f \in F\}$ and $\underline{d} = \min\{|F|: F \subset \omega^\omega$ and for each $g \in \omega^\omega$ there is an $f \in F$ with $g <^* f\}$. It is easily seen that, for any $p \in U(\omega)$, $\underline{b} \leq b(p) \leq \underline{d}$ and since it is consistent that $\underline{b} = \underline{d} = \kappa$ for any regular κ with $\omega_1 \leq \kappa \leq 2^\omega$, $b(p)$ cannot take the place of $\kappa(p)$ in 2.2. On the other hand it is a result of Rothberger that $\underline{b} = \min\{\kappa: P(\omega)/\text{fin}$ has an (ω,κ)-gap$\}$ and it is easily shown that $\underline{b} = \min\{\kappa: (\omega^\omega,<^*)$ has an (ω,κ)-gap$\}$ hence it is somewhat surprising that $\kappa(p)$ need not equal $\underline{b}$ or $b(p)$. However for P-points in $U(\omega)$ $\kappa(p) \geq \underline{b}$ (I do not know if $\kappa(p) = b(p)$). A point $p \in S(B)$, for a boolean algebra B, is a P_α-*point* if p is an α-complete filter, a P-point is a P_{ω_1}-point (i.e. if $A \in [p]^\omega$ then there is a $b \in p$ with $b < a$ for each $a \in A$).

2.4 *Proposition. If* $p \in U(\omega)$ *is a* P-*point then* $\underline{b} \leq \kappa(p) \leq \underline{d}$.

Proof. If $g \in \omega^\omega$ and $L(p,\underline{n},g)$ for each $n \in \omega$, then there is an $f \in \omega^\omega$ such that $E(p,g,f)$ while $f^{\leftarrow}(n)$ is finite for each $n \in \omega$. Now let $\{g_\alpha: \alpha < \kappa(p)\} \subset \omega^\omega$ be chosen so that $|g_\alpha^{\leftarrow}(n)| < \omega$ for each $n \in \omega$ and $(\{\underline{n}: n \in \omega\},\{g_\alpha: \alpha < \kappa(p)\})$ forms a gap in $(\omega^\omega,L(p))$. For each $\alpha < \kappa(p)$ and $n \in \omega$ define $f_\alpha(n) = \min\{k: g_\alpha(j) > n$ for $j \geq k\}$. We show that $\{f_\alpha: \alpha < \kappa(p)\}$ is unbounded in $(\omega^\omega,<^*)$. Indeed suppose that $f \in \omega^\omega$ is strictly increasing and $f_\alpha <^* f$ for

$\alpha < \kappa(p)$. Define $g(k) = \max\{n: f(n) \leq k\}$ for $k \in \omega$. Let $\alpha < \kappa(p)$ and choose $m \in \omega$ so that $f(n) > f_\alpha(n)$ for $n > m$. Now let $j > f(m)$ and let $g(j) = n$, hence $f_\alpha(n) < f(n) < j$ which means that $g_\alpha(j) > n$. Therefore $g <^* g_\alpha$ for all $\alpha < \kappa(p)$, which is a contradiction; and so $\kappa(p) \geq \underline{b}$. Now let $H \subset \omega^\omega$ be increasing functions with $|H| = \underline{d}$ so that for each $f \in \omega^\omega$ there is an $h \in H$ with $f <^* h$. For each $f \in H$, $|\{\alpha < \kappa(p): f_\alpha <^* f\}| < \kappa(p)$ since otherwise we could define g as above and have $g <^* g_\alpha$ for $\alpha < \kappa(p)$. Therefore, since, for each $\alpha < \kappa(p)$, there is an $h \in H$ with $f_\alpha <^* h$, $\kappa(p) \leq |H|$.

Before we can give the proofs of 2.1 and 2.2 we need some preliminary results.

2.5 *Definition.* Let $F \subset \alpha^\alpha$ and let p be a filter on α. F is of *large oscillation mod* p if for any $n < \omega$, $\{f_1,\cdots,f_n\} \subset F$, $(\gamma_1,\cdots,\gamma_n) \in \alpha^n$ and $A \in p$ the set $A \cap \cap\{f_i^{\leftarrow}(\gamma_n): 1 \leq i \leq n\}$ is not empty.

The above definition and the following result are in [EK].

2.6 *Theorem. There is a set* $F \subset \alpha^\alpha$ *of cardinality* 2^α *such that* F *is of large oscillation mod* p *where* $p = \{A \subset \alpha: |\alpha\backslash A| < \alpha\}$.

Kunen constructed α^+-good ultrafilters on α using the following idea.

2.7 *Lemma. Suppose that* p *is a filter on* α, $F \subset \alpha^\alpha$ *is of large oscillation mod* p, W *is a function from* $[\alpha]^{<\omega}$ *into*

p *and* A *is a subset of* α. *There is a filter* $p' \supset p$, *and* $F' \subset F$ *and a multiplicative function* V *from* $[\alpha]^{<\omega}$ *into* p' *so that* V *refines* W, $|F \backslash F'| < \omega$, *either* A *or* $\alpha \backslash A$ *is in* p' *and* F' *is of large oscillation mod* p'.

Proof. We first find V. Let $\{H_\gamma : \gamma < \alpha\}$ be a listing of $[\alpha]^{<\omega}$ and let $f_\emptyset \in F$ be arbitrary. For each $H \in [\alpha]^{<\omega}$, let $W'(H) = \cap\{W(J) : J \subseteq H\}$, and define $V(H) = \cup\{\overleftarrow{f_\emptyset}(\gamma) \cap W'(H_\gamma) : H \subset H_\gamma\}$. For each $\delta \in H$ and γ with $H \subset H_\gamma$, $V(\{\delta\}) \cap \overleftarrow{f_\emptyset}(\gamma) = W'(H_\gamma)$ and for γ with $H \backslash H_\gamma \neq \emptyset$ there is a $\delta \in H$ with $V(\{\delta\}) \cap \overleftarrow{f_\emptyset}(\gamma) = \emptyset$. It follows that V is multiplicative. Let $p_\emptyset$ be the filter generated by $p \cup \{V(\{\delta\}) : \delta < \alpha\}$; $p_\emptyset$ is a filter since for $D \in p$, $\gamma < \alpha$ $D \cap V(H_\gamma) \supset D \cap W'(H_\gamma) \cap \overleftarrow{f_\emptyset}(\gamma) \neq \emptyset$. It is routine to check that $F \backslash \{f_\emptyset\}$ is of large oscillation mod $p_\emptyset$. If $F \backslash \{f_\emptyset\}$ is of large oscillation mod the filter generated by $p_\emptyset \cup \{A\}$ then let these be F' and p' respectively. Otherwise there are $f_1, \cdots, f_n \in F \backslash \{f_\emptyset\}$, $(\gamma_1, \cdots, \gamma_n) \in \alpha^n$ and $D \in p_\emptyset$ with $D \cap A \cap \cap\{\overleftarrow{f_i}(\gamma_i) : i = 1, \cdots, n\} = \emptyset$. In this case we let $F' = F \backslash \{f_\emptyset, f_1, \cdots, f_n\}$ and let p' be the filter generated by $p_\emptyset \cup \{\overleftarrow{f_i}(\gamma_i) : i = 1, \cdots, n\}$.

The construction of an ω-incomplete α^+-good ultrafilter is then just an induction of length 2^α using 2.7 and being sure to introduce enough multiplicative functions and to make sure it is maximal. In order to prove 2.1 we simply add a few steps to the induction according to 2.8.

2.8 *Lemma*. *If* p *and* F *are as in* 2.7, $\omega_i \leq \alpha$ *and* $H = \{h \in \alpha^\alpha : L(p, \underline{\gamma}, h)$ *for all* $\gamma < \omega_i\}$ *then there is a* *filter* p' *and a function* $f \in F$ *so that* $L(p,\underline{\gamma},f,h)$ *for* $\gamma < \omega_i$, $h \in H$ *and* $F\setminus\{f\}$ *is of large oscillation mod* p'.

Proof. Let $f \in F$ be arbitrary and let p' be the filter generated by $p \cup \{\cup\{f^{\leftarrow}(\delta) \cap h^{\leftarrow}((\delta,\omega_i)): \gamma < \delta < \omega_i\}: \gamma < \omega_i$ and $h \in H\}$. We show that $F\setminus\{f\}$ is of large oscillation mod p'. Indeed suppose that $f_1,\cdots,f_n \in F\setminus\{f\}$, $(\gamma_1,\cdots,\gamma_n) \in \alpha^n$, $A \in p$, $\gamma < \omega_i$ and $h \in H$ (note that H is closed under finite meets). Let $\gamma < \delta < \omega_i$, then $A \cap h^{\leftarrow}((\delta,\omega_i)) = A' \in p$ since $L(p,\underline{\gamma},h)$. Therefore $A' \cap f^{\leftarrow}(\delta) \cap \cap\{f_j^{\leftarrow}(\gamma_j): j = 1,\cdots,n\} \neq \emptyset$. It is clear that, for $\gamma < \omega_i$, $L(p,\underline{\gamma},f)$ and, for $h \in H$, $\{j < \alpha: f(j) < h(j)\} \supset \cup\{f^{\leftarrow}(\delta) \cap h^{\leftarrow}((\delta,\omega_i)): \delta < \omega_i\} \in p'$.

Proof of Theorem 2.1. Starting with a family F given in 2.6 perform an induction of length 2^α to construct a chain of filters $\{p_\delta: \delta < 2^\alpha\}$ using, for instance, 2.8 when $cf(\delta) = \omega_i$ and 2.7 otherwise. To see that, for $\omega_i \le \alpha$ with ω_i regular, $\kappa(i,p) \ge cf(2^\alpha)$ observe that if $H \subset \alpha^\alpha$, $|H| < cf(2^\alpha)$ and $L(p,\underline{\gamma},h)$ for $\gamma < \omega_i$ and $h \in H$ then there is some $\delta < 2^\alpha$ with $cf(\delta) = \omega_i$ such that $L(p_\delta,\underline{\gamma},h)$ for $\gamma < \omega_i$ and $h \in H$. Therefore by 2.8, there is an $f \in \alpha^\alpha$ with $L(p_{\delta+1},\underline{\gamma},f,h)$ for $\gamma < \omega_i$, $h \in H$. Also, if $D \subset 2^\alpha$ is cofinal with $cf(\delta) = \omega_i$ for $\delta \in D$, then there are f_δ, $\delta \in D$, so that if $L(p,\underline{\gamma},h)$ then $L(p_\delta,\underline{\gamma},h)$ for some $\delta \in D$ and so $L(p,\underline{\gamma},f_\delta,h)$.

Proof of Theorem 2.2. In this case $\alpha = \omega$ and so we do not have to worry about making the filter α^+-good. Let κ be any regular cardinal with $\omega_1 \le \kappa \le 2^\omega$ and let $F \subset \omega^\omega$ be of large oscillation mod the cofinite filter with $|F| = \kappa$. For each $f \in F$, let g_f be the map from $U(\omega)$ onto the ordinal space $\omega + 1$ defined by $g_f^{\leftarrow}(n) = [f^{\leftarrow}(n)]^*$ and $g_f^{\leftarrow}(\omega) = U(\omega)\setminus \cup\{g_f^{\leftarrow}(n): n \in \omega\}$. Now let G be the map from $U(\omega)$ onto $(\omega + 1)^F$ which is just the product of the g_f's, $f \in F$. Finally, using a Zorn's Lemma argument, we find a closed set $K \subset U(\omega)$ so that G maps K onto $(\omega + 1)^F$ but no proper closed subset of K maps onto $(\omega + 1)^F$. Let $p_\emptyset = \{A \subset \omega: K \subset A^*\}$ and note that F is of large oscillation mod $p_\emptyset$ since $K \cap \cap\{f_i^{\leftarrow}(n_i)^*: i = 1,\cdots,n\} \neq \emptyset$ for all $\{f_1,\cdots,f_n\} \subset F$ and $n_i \in \omega$. The following Fact is the key to the whole proof. Let $F = \{f_\alpha: \alpha < \kappa\}$ and let $X = (\omega + 1)^F$.

Fact 1. *If* $A \subset \omega$ *then there is a countable set,* $\mathrm{supp}(A) \subset \kappa$ *such that if* $x \in G(A^* \cap K)$ *and* $y \in X$ *with* $y(f_\alpha) = x(f_\alpha)$ *for* $\alpha \in \mathrm{supp}(A)$ *then* $y \in G(A^* \cap K)$, *and* $\mathrm{supp}(A)$ *is minimal with respect to this property.*

Proof of Fact 1. Let $S = \cup\{\omega^H: H \in [F]^{<\omega}\}$ and for $s \in S$ let $[s]$ be the clopen subset of X given by $[s] = \{x \in X: s \subset x\}$. Recall that each non-empty open subset of X contains an element of $S' = \{[s]: s \in S\}$ and that any set of pairwise disjoint members of S' is countable. Now, for $A \subset \omega$, choose $T \in [S]^{\leq\omega}$ so that $T' = \{[t]: t \in T\}$ is a maximal collection of pairwise disjoint clopen subsets of $G(A^* \cap K)$. Clearly, $\cup T'$ is dense in $X\setminus G(K\setminus A^*)$. Therefore $G(G^{\leftarrow}(\overline{\cup T'}) \cap K) \cup G(K\setminus A^*) = X$ and since $(G^{\leftarrow}(\overline{\cup T'}) \cap K) \cup K\setminus A^*$ is closed, it follows that $G(A^* \cap K) = \overline{\cup T'}$. Let $\mathrm{supp}_T(A) = \{\alpha: f_\alpha \in \cup\{t: t \in T\}\}$. Now since $x \in G(A^* \cap K)$ if and only if $x \in \overline{\cup T'}$ the proof of Fact 1 is complete if we can find a minimal supp(A). Indeed supp(A) = $\{\alpha: \exists s \in S$ and $n < \omega$ such that $[s] \not\subset G(A^* \cap K)$ and $[s \cup (f_\alpha, n)] \subset G(A^* \cap K)\}$. By definition $[t\restriction_{\text{supp}(A)}] \subset G(A^* \cap K)$ for each $t \in T$, so it suffices to show that supp(A) $\subset$ $\text{supp}_T(A)$. Suppose $\alpha \in$ supp(A) and s, n exhibit this fact. Then let $y \in [s]\setminus G(A^* \cap K)$ and let $x(f_\beta) = y(f_\beta)$ for $\beta \neq \alpha$ and $x(f_\alpha) = n$. Since $[s \cup (f_\alpha, n)] \subset G(A^* \cap K)$, $x \in G(A^* \cap K)$. Since $\text{supp}_T(A)$ has the first property stated in Fact 1 it follows that supp(A) $\subset$ $\text{supp}_T(A)$ and we are done.

We define a chain of filters $\{p_\alpha: \alpha < \kappa\}$ so that if supp(A) $\subset \alpha$ then A or $\omega\setminus A$ is in $p_{\alpha+1}$, if $A \in p_\alpha$ then supp(A) $\subset \alpha$ and $\{f_\delta: \delta \geq \alpha\}$ is of large oscillation mod p_α. Suppose $\alpha < \kappa$ and we have defined $\{p_\gamma: \gamma < \alpha\}$. If α is a limit then let $p_\alpha = \cup\{p_\gamma: \gamma < \alpha\}$. Now suppose that $\alpha = \gamma + 1$ and let $H_\gamma = \{h \in \omega^\omega: L(p_\gamma, \underline{n}, h)$ for $n < \omega\}$. Just as in 2.8, let p'_γ be the filter generated by $p_\gamma \cup \{\cup\{f_\gamma^{\leftarrow}(n) \cap h^{\leftarrow}((n,\omega)): n > m\}: m \in \omega, h \in H_\gamma\}$. Extend p'_γ to a filter p_α maximal with respect to the property that $A \in p_\alpha$ implies supp(A) $\subset \alpha$.

Let us check that $\{f_\delta: \delta \geq \alpha\}$ is of large oscillation mod p_α. First of all, by the minimality of supp(A) for $A \subset \omega$, it is clear that supp(A) $\subset \alpha$ for $A \in p'_\gamma$. Now if $A \in p_\alpha$, then supp(A) $\subset \alpha$ and also $G(K \cap A^*) \neq \emptyset$ because $p'_\gamma \supset p_\emptyset$. Choose $x \in G(K \cap A^*)$ and let $\{\delta_i: i = 1,\cdots,n\} \subset \kappa\setminus\gamma$ and $n_i \in \omega$ $i = i,\cdots,n$. Let $y \in X$ be defined so that $y(f_{\delta_i}) = n_i$ for $i = 1,\cdots,n$ and $y(f_\gamma) = x(f_\gamma)$ for $\gamma < \alpha$. By Fact 1, $y \in G(K \cap A^*)$ and clearly $y \in G(\cap\{f_{\delta_i}^{\leftarrow}(n_i)^*:$

$i = 1,\cdots,n\} \cap K)$. Therefore $A^* \cap \cap\{f^{\leftarrow}_{\delta_i}(n_i)^*: i = 1,\cdots,n\} \neq \emptyset$ since $\cap\{f^{\leftarrow}_{\delta_i}(n_i)^*: i = 1,\cdots,n\} \supset G^{\leftarrow}(y)$.

Finally we must show that if $p = \cup\{p_\alpha: \alpha < \kappa\}$ then $\kappa(p) = \kappa$. Indeed, let $H \subset \omega^\omega$ with $|H| < \kappa$ and suppose that $L(p,\underline{n},h)$ for each $n \in \omega$ and $h \in H$. Let $\gamma < \kappa$ be large enough so that for each $n \in \omega$, $h \in H$, $\mathrm{supp}(h^{\leftarrow}(n,\omega)) \subset \gamma$. Therefore $H \subset H_\gamma$ and by our construction $L(p,\underline{n},f_{\gamma+1},h)$ for each $n \in \omega$ and $h \in H$. Therefore $\kappa(p) = \kappa$.

As mentioned above Roitman proved that 2.2 holds consistently. In fact her techniques can be used to prove much more; it is consistent that B^ω/p can be $\underline{c}$-saturated providing that $B = CO(2^\omega)$.

2.9 *Theorem* [R]. *If* M *is a model obtained by adding* ω_2 *Cohen reals to a model of* $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, *then there is a* $p \in U(\omega)$ *such that* $[CO(2^\omega)]^\omega/p$ *is* ω_2*-saturated.*

This is also a theorem of MA (Martin's Axiom) and even $P(\underline{c})$. $P(\underline{c})$ holds if for each free filter p on ω with $|p| < \underline{c}$ there is an infinite $A \subset \omega$ so that $|A\backslash D| < \omega$ for $D \in p$.

2.10 *Theorem.* $(P(\underline{c}))$ *There is a point* $p \in U(\omega)$ *so that* $[CO(2^\omega)]^\omega/p$ *is* $\underline{c}$*-saturated. Furthermore* p *can be chosen to be a* $P_{\underline{c}}$*-point.*

Proof. $P(\underline{c})$ implies that $2^\kappa = \underline{c}$ for each $\kappa < \underline{c}$ and so we choose a listing $\{(F_\gamma,G_\gamma): \gamma < \underline{c}\}$ of all pairs of subsets of size less than $\underline{c}$ of $[CO(2^\omega)]^\omega$ so that each pair appears $\underline{c}$ times. Construct a chain of filters on ω,

$\{p_\gamma: \gamma < \underline{c}\}$, so that $|p_\gamma| \leq \omega \cdot |\gamma|$ as follows. We set $p_\emptyset = \emptyset$, p_1 = cofinite. At limits we take unions and at successor steps we ensure that if $F_\gamma \cup G_\gamma$ is a chain under $L(p)$ and $L(p,F_\gamma,G_\gamma)$ then there is an $h \in B^\omega$ with $L(p_{\gamma+1},F_\gamma,h,G_\gamma)$ where $B = \{b_m: m \in \omega\} = CO(2^\omega)\setminus\{\emptyset\}$. Indeed, for $A \in p_\gamma$, $f \in F_\gamma$ and $g \in G_\gamma$, let $A_{f,g} = \{(k,m): k \in A,$ $f(k) < b_m < g(k)\}$. If $L(p,F_\gamma,G_\gamma)$, then $q_\gamma = \{A_{f,g}: A \in p,$ $f \in F_\gamma$, $g \in G_\gamma\}$ is a filter base of cardinality less than $\underline{c}$. By $P(\underline{c})$, we choose $C \subset \omega \times \omega$ such that $|C\setminus A_{f,g}| < \omega$ for each $A_{f,g} \in q_\gamma$. Now since C is infinite and p_γ contains the cofinite filter, $D = \{k: C \cap \{k\} \times \omega \neq \emptyset\}$ is infinite. Define $h \in B^\omega$ so that, for $k \in D$, $h(k) = b_m$ implies $m \in C$. Now if we let $p_{\gamma+1}$ be the filter generated by $p_\gamma \cup \{D\}$ then $\{k \in D: f(k) \nless h(k) \text{ or } h(k) \nless g(k)\} \subset \{k: C\setminus A_{f,g} \cap \{k\} \times \omega \neq \emptyset\}$ and so is finite. Also $D\setminus A$ is finite for each $A \in p_\gamma$ hence $p = \cup p_\gamma$ is a $P_{\underline{c}}$-point. Now B^ω/p has no (κ,λ)-gaps for $\kappa,\lambda < \underline{c}$ and by a result in [D] this ensures that it is $\underline{c}$-saturated.

3. Applications to Boolean Algebras and Topology

If B is an atomless boolean algebra and $p \in U(\omega)$, it follows from 1.1 that B^ω/p is an ω_1-saturated boolean algebra. It is well known that $P(\omega)/\text{fin}$ is ω_1-saturated and so it is natural to be interested in determining which properties B^ω/p and $P(\omega)/\text{fin}$ share and which they need not. In particular Balcar and Vojtas showed that each ultrafilter of $P(\omega)/\text{fin}$ has a disjoint refinement and asked for which other algebras is this true. Also van Douwen showed that this and some other properties of $P(\omega)/\text{fin}$ are shared by those ω_1-saturated boolean algebras of cardinality $\underline{c}$ whose Stone spaces map onto $U(\omega)$ by an open map.

A point x in a space X is called a *κ-point* for a cardinal κ if there are κ disjoint open subsets of X such that x is in the closure of each. If $X = S(B)$ where B is an α^+-saturated boolean algebra and $\kappa = 2^\alpha$, then this is equivalent to the corresponding ultrafilter of B having a *disjoint refinement* (that is, there is a function f from p $S(B)$ to $B\backslash\{0\}$ such that $f(b) < b$ and $f(b) \wedge f(c) = 0$ for $b,c \in p$). A subset $\{b(i,j): (i,j) \in I \times J\}$ of B is called an $I \times J$-*independent matrix* if $b(i,j) \wedge b(i,j') = 0$ and $\wedge\{b(i,f(i)): i \in I'\} \neq 0$ for any $i \in I' \in [I]^{<\omega}$, $f \in J^{I'}$ and $j \neq j' \in J$. B has an $I \times J$-independent matrix if and only if $S(B)$ maps onto $(D(J) + 1)^I$ where $D(J) + 1)^I$ has the product topology and $D(J) + 1$ is the one point compactification of the discrete space J. Kunen introduced independent matrices in [K2], he showed that $P(\omega)/\text{fin}$ has a $2^\omega \times 2^\omega$-independent matrix and used this to construct 2^ω-OK points. As mentioned above Balcar and Vojtas [BV] showed that every point of $U(\omega)$ is a 2^ω-point.

3.1 *Theorem* [vD]. *Let* B *be an ω_1-saturated boolean algebra with* $|B| = 2^\omega$ *such that* $S(B)$ *maps onto* $U(\omega)$ *by an open map. (For example see the end of section 1).*

(0) $S(B)$ *has* P-*points if and only if* $U(\omega)$ *has* P-*points.*

(1) B *has a* $2^\omega \times 2^\omega$-*independent matrix.*

(2) *Every point of* $S(B)$ *is a* 2^ω-*point.*

(3) *If* $P(\omega)/\mathrm{fin}$ *has an* (ω,λ)*-gap then so does* B. *(In particular* B *has an* $(\omega,\underline{b})$*-gap and it is consistent that* $\underline{b} < \lambda$*).*

Now let α be an infinite cardinal and let B be any atomless boolean algebra with $|B| \leq 2^\alpha$. Also let p be an ω-incomplete α^+-good ultrafilter on α.

3.2 *Theorem.* (0) $S(B^\alpha/p)$ *has a dense set of* P_{α^+}*-points.*

(1) B^α/p *has a* $2^\alpha \times 2^\alpha$*-independent matrix.*

(2) *Each point of* $S(B^\alpha/p)$ *is a* 2^α*-point.*

(3) B^α/p *has an* (ω_i,κ)*-gap if and only if* $\kappa = \kappa(i,p)$ *for each regular* $\omega_i \leq \alpha$.

3.2 (0) *Proof.* Let $f \in (B\backslash\{0\})^\alpha$ and for each $\gamma < \alpha$ choose $y_\gamma \in S(B)$ so that $f(\gamma) \in y_\gamma$. We show that $x = \{g \in B^\alpha/p\colon g(\gamma) \in y_\gamma \text{ for } \gamma \in \alpha\}$ is a P_{α^+}-point of $S(B^\alpha/p)$. Indeed, let $\{g_\delta\colon \delta < \alpha\} \subset x$ and $\{A_n\colon n \in \omega\} \subset p$ so that $\cap A_n = \emptyset$. Define $W\colon [\alpha]^{<\omega} \to p$ by $W(H) = A_{|H|} \cap \{\gamma < \alpha\colon g_\delta(\gamma) \in y_\gamma \text{ for } \delta \in H\}$. Now let $V\colon [\alpha]^{<\omega} \to p$ be a multiplicative function refining W. As usual, for each $\gamma \in \alpha$, $H_\gamma = \{\delta \in \alpha\colon \gamma \in V(\{\delta\})\}$ is finite. Also, since $V(H_\gamma) \subset W(H_\gamma)$ and B is atomless we may choose $g(\gamma) \in y_\gamma$ so that $g(\gamma) < g_\delta(\gamma)$ for $\delta \in H_\gamma$. It follows that $g \in x$ and that $L(p,g,g_\delta)$ for each $\delta < \alpha$.

3.2 (1) *Proof.* Since B is atomless we may choose $\{b(n,m)\colon n,m \in \omega\} \subset B$ to be an $\omega \times \omega$-independent matrix (i.e. S(B) maps onto $(\omega + 1)^\omega$). For each $f,g \in \omega^\alpha/p$ define $a_{fg} \in B^\alpha$ by $a_{fg}(\gamma) = b(f(\gamma),g(\gamma))$. We verify that

$\{a_{fg}$: $f,g \in \omega^\alpha/p\}$ is an independent matrix. Indeed, if $f,g,h \in \omega^\alpha$ with $L(p,g,h)$ then $\{\gamma \in \alpha: a_{fg}(\gamma) \wedge a_{fh}(\gamma) = 0\} = \{\gamma \in \alpha: b(f(\gamma),g(\gamma)) \wedge b(f(\gamma),h(\gamma)) = 0\} = \{\gamma \in \alpha: g(\gamma) \neq h(\gamma)\} \in p$. Similarly if F is a finite subset of ω^α/p and G is a function from F into ω^α/p then $\{\gamma \in \alpha: \wedge\{a_{f,G(f)}(\gamma): f \in F\} \neq 0\} \supset \{\gamma \in \alpha: \wedge\{b(f(\gamma),G(f)(\gamma)): f \in F \neq 0\} \supset \{\gamma \in \alpha: |\{f(\gamma): f \in F\}| = |F|\} \in p$.

Before we prove 3.2(2) we prove a result which is proven about $P(\omega)/\text{fin}$ in [BV] although it is not stated explicitly.

3.3 *Lemma. If* $\lambda \leq \alpha$ *and* $\{a_\eta: \eta < \lambda\} \subset B^\alpha/p$ *with* $a_\eta \wedge a_\xi = 0$ *for* $\eta < \xi < \lambda$ *then the set* $C = \{b \in B^\alpha/p$: $\{\eta: b \wedge a_\eta \neq 0\}$ *is infinite has a disjoint refinement.*

Proof. Let $\{A_m: m \in \omega\} \subset p$ with $\cap A_m = \emptyset$ and for $H \in [\lambda]^{<\omega}$ define $W(H) = \{\gamma \in \alpha: a_\eta(\gamma) \neq 0$ and $a_\eta(\gamma) \wedge a_\xi(\gamma) = 0$ for $n \neq \xi$ and $n,\xi \in H\} \cap A_{|H|}$. Let V be a multiplicative map from $[\lambda]^{<\omega}$ to p which refines W. Let $C = \{c_\delta$: $\delta \in 2^\alpha\}$ and define $I_\delta = \{\eta \in \lambda: L(p,0,c_\delta \wedge a_\eta)\}$. Also let $H_\gamma = \{\eta \in \lambda: \gamma \in V(\{\eta\})\}$ and define $S_\gamma^\delta = \{a_\eta(\gamma): \eta \in H_\gamma \cap I_\delta$ and $a_\eta(\gamma) \wedge c_\delta(\gamma) \neq 0\}$ (and $S_\gamma^\delta = \{\emptyset\}$ if this is empty) for each $\gamma < \alpha$ and $\delta < 2^\alpha$. Now if $H \in [I_\delta]^{<\omega}$, $\{\gamma \in \alpha$: $|S_\gamma^\delta| > |H|\} \supset V(H) \cap \{\gamma \in \alpha: c_\delta(\gamma) \wedge a_\eta(\gamma) \neq 0$ for $\gamma \in H\} \in p$. Therefore, by 1.2, $|\Pi_{\gamma<\alpha}S_\gamma^\delta/p| = 2^\alpha$ for each $\delta \in 2^\alpha$. It follows, therefore, that for $\delta \in 2^\alpha$, we may choose $d_\delta \in \Pi_{\gamma<\alpha}S_\gamma^\delta/p$ so that $E(p,0,d_\delta \wedge a_\eta)$ for $\eta < \lambda$ and $\neg E(p,d_\delta,d_\beta)$ for $\beta < \delta < 2^\alpha$. Now let $\beta < \delta < 2^\alpha$, we show that $E(p,0,d_\delta \wedge d_\beta)$. Indeed, let $\eta_0 \in I_\beta$ and $\eta_1 \in I_\delta$ be arbitrary and let $\gamma \in V(\{\eta_0\}) \cap V(\{\eta_1\}) \cap \{\gamma \in \alpha: d_\beta(\gamma) \neq d_\delta(\gamma)\} \in p$. Now, by choice of γ, if $d_\delta(\gamma) = a_\eta(\gamma)$ and $d_\beta(\gamma) = a_\xi(\gamma)$ then $\{\eta,\xi\} \subset H_\gamma$ and so $\gamma \in V(\{\eta,\xi\}) \subset W(\{\eta,\xi\})$ which implies $a_\eta(\gamma) \wedge a_\xi(\gamma) = 0$. Therefore, for $\delta < 2^\alpha$ and $\gamma < \alpha$, let $e_\delta(\gamma) = d_\delta(\gamma) \wedge c_\delta(\gamma)$ and we have our disjoint refinement.

Similarly one can prove that if $\{a_\eta: \eta < \lambda\} \subset B^\alpha/p$ is an increasing chain (with λ a limit) then $C = \{b \in B^\alpha/p: \{\eta: b \wedge a_\eta - a_\xi \neq 0 \text{ for } \xi < \eta\}$ is cofinal in $\lambda\}$ has a disjoint refinement.

3.2 (2) *Proof.* Let $x \in S(B^\alpha/p)$ and suppose that $\{a_\eta: \eta < \lambda\} \subset B^\alpha/p$ is chosen with λ minimal such that $\{a_\eta: \eta < \lambda\}$ is an increasing chain, $x \notin \{a_\eta{}^*: \eta < \lambda\}$ (i.e. $a_\eta \notin x$ for $\eta < \lambda$) and for $a \in x$ there is an $\eta < \lambda$ with $a \wedge a_\eta \neq 0$ (i.e. $x \in \mathrm{cl} \cup a_\eta{}^*$). Let $a_\lambda = 1$ and for each $\gamma \leq \lambda$ with $\mathrm{cf}(\gamma) = \omega$ let $C_\gamma = \{b \in B^\alpha/p: b \leq a_\gamma$ and $\{\eta \leq \gamma: b \wedge a_\eta - a_\xi \neq 0 \text{ for } \xi < \eta\}$ is cofinal in $\gamma\}$. By Lemma 3.3 (with $\lambda = \omega$), the set C_γ has a disjoint refinement C'_γ so that for $c \in C'_\gamma$, $c \leq a_\gamma - a_\eta$ for $\eta < \gamma$. Therefore $\cup\{C'_\gamma: \gamma \leq \lambda \text{ with } \mathrm{cf}(\gamma) = \omega\}$ is a disjoint refinement of $\cup\{C_\gamma: \gamma \leq \lambda, \mathrm{cf}(\gamma) = \omega\}$. To complete the proof it suffices to show that for $a \in x$ there is a $\gamma \leq \lambda$ with $\mathrm{cf}(\gamma) = \omega$ and $a \wedge a_\gamma \in C_\gamma$. Indeed choose $\gamma_0 < \lambda$ so that $a \wedge a_{\gamma_0} \neq 0$, if we have chosen $\gamma_n < \lambda$ choose $\gamma_{n+1} < \lambda$ so that $a - a_{\gamma_n} \wedge a_{\gamma_{n+1}} \neq 0$. Now if $\gamma = \sup\{\gamma_n: n \in \omega\}$ we have that $a \wedge a_\gamma \in C_\gamma$.

3.2 (3) *Proof.* This is just 1.4.

3.4 *Corollary.* $2^\omega > \omega_1$ *implies there are* $p,q \in U(\omega)$ *so that* $[CO(2^\omega)]^\omega/p \not\cong [CO(2^\omega)]^\omega/q$ *and* $S([CO(2^\omega)]^\omega/p)$ *does not map onto* $U(\omega)$ *by an open map.*

Proof. This follows from 2.2, 3.1(3) and 3.2(3).

Let $B = CO(2^\omega)$ and let M be the model of set theory described in 2.9. Kunen has shown that in this model $P(\omega)/\text{fin}$ has no chains of order type ω_2. However if we let $p \in U(\omega)$ be chosen so that B^ω/p is ω_2-saturated as in 2.9 we have the following result.

3.5 *Proposition. It is consistent that there is a* $p \in U(\omega)$ *such that* $P(\omega)/\text{fin}$ *embeds into* B^ω/p *but* B^ω/p *does not embed into* $P(\omega)/\text{fin}$. *Equivalently* $S(B^\omega/p)$ *maps onto* $U(\omega)$ *but* $U(\omega)$ *does not map onto* $S(B^\omega/p)$.

In [BFM], the authors introduce a condition which they call (*) where (*) is the statement "each closed subset of $U(\omega)$ is homeomorphic to a nowhere dense $P_{\underline{c}}$-set of $U(\omega)$." They show that CH implies (*) and that MA + $\underline{c} = \omega_3$ implies (*) is false. We verify their conjecture that (*) implies CH. A subset of K of a space X is a P_α*-set* if the filter of neighborhoods of K is α-complete (K is a P-set if it is a P_{ω_1}-set).

3.6 *Lemma. If* $K \subset U(\omega)$ *is a closed* P_α*-set and for some* κ,λ *with* $\omega \leq \kappa \leq \alpha$ *and* $\omega \leq \lambda$, $CO(K)$ *has a* (κ,λ)*-gap then* $CO(U(\omega))$ *has a* (κ,λ')*-gap for some* $\omega \leq \lambda' \leq \lambda$.

Proof. Let $\{a_\gamma: \gamma < \kappa\} \cup \{b_\beta: \beta < \lambda\} \subset CO(K)$ so that $\gamma_1 < \gamma_2 < \kappa$ and $\beta_1 < \beta_2 < \lambda$ implies $a_{\gamma_1} < a_{\gamma_2} < b_{\beta_2} < b_{\beta_1}$. Choose $\{a'_\gamma: \gamma < \kappa\} \subset CO(U(\omega))$ so that $a'_\gamma \cap K = a_\gamma$ for $\gamma < \kappa$. For each $\gamma < \kappa$, we can find $U_\gamma \in CO(U(\omega))$ so that $K \subset U_\gamma$ and $U_\gamma \cap a'_\gamma - a'_\delta = \emptyset$ for $\delta < \gamma$. Also since $\kappa < \alpha$, there is a U in $CO(U(\omega))$ with $K \subset U$ so that $U \subset U_\gamma$ for $\gamma < \kappa$. Therefore we may suppose that $a'_\delta \subset a'_\gamma$ for $\delta < \gamma < \kappa$. Now, choose for as long as possible, $b'_\beta \in CO(U(\omega))$ so that $b'_\beta \cap K = b_\beta$ and $a'_\gamma \subset b'_\beta \subset b'_\delta$ for $\delta < \beta$ and $\gamma < \kappa$. Therefore, for some $\lambda' \leq \lambda$, we cannot choose $b'_{\lambda'}$, and we have a gap in $CO(U(\omega))$.

3.7 *Proposition. If* $\beta\omega$ *embeds into* $U(\omega)$ *as a* P_α*-set then* $\underline{b} \geq \alpha$.

Proof. Suppose that $\{p_n: n \in \omega\}$ is a discrete subset of $U(\omega)$ such that $K = cl_{\beta\omega}\{p_n: n \in \omega\}$ is a P_α-set (it is well known that K is homeomorphic to $\beta\omega$). Choose pairwise disjoint subsets $\{A_n: n \in \omega\}$ of ω so that $A_n \in p_n$, and fix an indexing $A_n = \{a(n,m): m \in \omega\}$ for each $n \in \omega$. Let $F \subset \omega^\omega$ with $|F| < \alpha$; we show that F is bounded. For each $f \in F$, let $B_f = \{a(n,m): n \in \omega \text{ and } m > f(n)\}$. Clearly $K \subset B_f^*$ for $f \in F$ and so we may choose $B \subset \omega$ so that $K \subset B^*$ and $|B \backslash B_f| < \omega$ for $f \in F$. Let $g \in \omega^\omega$ be defined by $g(n) = \min\{m: a(n,m) \in B\}$ and observe that $f <^* g$ for $f \in F$.

3.8 *Theorem*. (*) *is equivalent to* CH.

Proof. Clearly if (*) is true then $\beta\omega$ must embed in $U(\omega)$ as a $P_{\underline{c}}$-set. Therefore by 3.7 it suffices to show that $\underline{b} = \omega_1$. Now let $p \in U(\omega)$ be chosen so that $\kappa(p) = \omega_1$. Let $\{a_n: n \in \omega\} \subset CO(U(\omega))$ be pairwise disjoint and let

$K^p = \cap\{cl_{U(\omega)} \cup \{a_n: n \in A\}: A \in p\}$. It is well known that $cl_{U(\omega)} \cup \{a_n: n \in \omega\} \cong \beta(\omega \times U(\omega))$. Therefore $CO(K^p) \cong [CO(U(\omega))]^{\omega}/p$ and by 1.4 has an (ω,ω_1)-gap. By 3.6, if K^p embeds into $U(\omega)$ as a $P_{\underline{c}}$-set with $\underline{c} > \omega_1$ then $\underline{b} = \omega_1$.

3.9 *Remark*. It is not difficult to show that if A is a boolean algebra which has an (ω_1,ω_1)-gap then so does A^{ω}/p for each $p \in U(\omega)$ and is therefore not ω_2-saturated. This means that we cannot easily obtain compact subsets K of $U(\omega)$ so that $CO(K)$ is ω_2-saturated (such as subsets of the boundary of a cozero set). However $S(B^{\omega}/p) = K^p$ as in 3.5 is in some sense a "well-placed" subset of $\beta(\omega \times 2^{\omega})$. For instance K^p is a 2^{ω}-set in $(\omega \times 2^{\omega})^* = \beta(\omega \times 2^{\omega})\setminus(\omega \times 2^{\omega})$ (see [BV]). Furthermore we can easily construct p to be 2^{ω}-OK (see [K2]) in which case every ccc subspace of $(\omega \times 2^{\omega})^*$ meets K^p in a nowhere dense set. Furthermore if we use 2.10 to find p a $P_{\underline{c}}$-point then K^p is a $P_{\underline{c}}$-set in $(\omega \times 2^{\omega})^*$. I do not know if it is possible to find a P_{ω_2}-set K in $U(\omega)$ such that $CO(K)$ is ω_2-saturated. Although Shelah has found a model in which $U(\omega)$ is not homeomorphic to $(\omega \times 2^{\omega})^*$ (see [vM]) it would be interesting if they were not in one of the above models.

After acceptance of this paper, John Merrill brought it to the author's attention that 2.2 and a more general version of 2.3 appear in Shelah's Model Theory book. However as the proofs presented here seem simpler we have chosen to include them.

References

[ACCH] M. Y. Antonovskij, D. V. Chudnovsky, G. V. Chudnovsky and E. Hewitt, *Rings of continuous real-valued functions II*, Math. Zeit. 176 (1981), 151-186.

[BFM] B. Balcar, R. Frankiewicz and C. F. Mills, *More on nowhere dense closed* P-*sets*, Bull. l'Acad. Pol. Sci. 28 (1980), 295-299.

[BV] B. Balcar and P. Vojtas, *Almost disjoint refinements of families of subsets of* N, Proc. AMS 79 (1980), 465-470.

[CN] W. W. Comfort and S. Negrepontis, *The theory of ultrafilters*, Springer, Berlin, New York (1974).

[vD] E. K. van Douwen, *Transfer of information about* $\beta N-N$ *via open remainder maps* (preprint).

[D] A. Dow, *Saturated Boolean algebras and their Stone spaces* (preprint). to appear in Top. and its Appl.

[EK] R. Engelking and M. Karlowicz, *Some theorems of set theory and their topological consequences*, Fund. Math. 57 (1965), 275-285.

[Ke] H. J. Keisler, *Good ideals in fields of sets*, Ann. Math. (2) 89 (1964), 338-359.

[K1] K. Kunen, *Ultrafilters and independent sets*, Trans. AMS 172 (1972), 299-306.

[K2] ________, *Weak* P-*points in* $\beta N-N$, Proc. Bolyai Janos Soc., Coll. on Top. Budapest (1978), 741-749.

[vM] J. van Mill, *An introduction to* $\beta\omega$, Handbook of set-theoretic topology, Eds. K. Kunen and J. E. Vaughan, North Holland, Amsterdam, (1984).

[R] J. Roitman, *Non-isomorphic* H-*fields from non-isomorphic ultrapowers*, (to appear), Math. Zeit.

University of Toronto

Toronto, Ontario, Canada M5S 1A1