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In this note, we construct a Baire space X, which, when given the G_δ-topology, is first category in itself. The space X will be a subspace of a product of two spaces, one of which is first countable and the other of which is a P-space. Therefore, the G_δ-topology will give the topological union of subspaces of X.

An ordinal is the set of its predecessors and a cardinal is an initial ordinal. We will denote by $\mathbf{M}$ the minimum cardinal of a dense, Baire subset of the space $\mathbb{R}$ of real numbers. Clearly, $\mathbf{M}$ has uncountable cofinality. If A is a subset of $\mathbb{R}$ which is second category in $\mathbb{R}$, then A has cardinality at least $\mathbf{M}$; for if card(A) $< \mathbf{M}$, then the union of all rational translates of A would be a dense Baire subset of $\mathbb{R}$ having cardinality less than $\mathbf{M}$. Also if D is a dense Baire subset of $\mathbb{R}$ and I is any non-empty open interval, then card$(I \cap M) \geq \mathbf{M}$.

If A and B are sets, ${}^{A}B$ denotes the set of functions from A to B. For each ordinal $\alpha \leq m$, let $Y_\alpha = \cup_{\beta<\alpha} {}^{\beta}\omega_1$ ordered by extension: $f \leq g$ if $f \subseteq g$. With this partial order, Y_α is a tree. For $p \in Y_\alpha$, let S(p) be the set of immediate successors of p, and for $K \subseteq S(p)$, let $N_K^\alpha(p) = \{p\} \cup \{x \in Y_\alpha : k \leq x \text{ for some } k \in K\}$. Topologize Y_α by using for a base $\{N_K^\alpha(p) : p \in Y_\alpha$ and $S(p)\backslash K$ is countable$\}$. If $\alpha < \beta$, then Y_α is a subspace of Y_β.

We will need the folloiwng lemma, the proof of which is given in [L].

Lemma 1. *Let* α *be a limit ordinal,* $\alpha \leq \mathbf{M}$.

(i) Y_α *is a regular* P*-space without isolated points.*

(ii) *If* α *has countable cofinality, then* Y_α *is first category in itself.*

(iii) *If* α *has uncountable cofinality, then* Y_α *is Baire. In particular, the Baire* P*-space* $Y_{\mathbf{M}}$ *is the union of the* $\mathbf{M}$ *subspaces,* $Y_{\alpha+\omega_0}$, $\alpha < \mathbf{M}$, *each of which is first category in itself.*

If X is a space, δX denotes X with the G_δ-topology, that is, the topology generated by the G_δ-sets of X. Before giving the main example, we give an easier example.

Example 1. Let $X = \mathbf{R} \times Y_{\omega_0}$ with the usual product topology strengthened so that if $u \neq 0$, the point (u,v) is isolated. Then X has a dense set of isolated points, so it is Baire, but δX contains $\{0\} \times Y_{\omega_0}$ as an open subspace, and $\{0\} \times Y_{\omega_0}$, being homeomorphic to Y_{ω_0}, is first category in itself by Lemma 1. Thus, δX is not Baire.

The idea of the main example is to make certain that the pathology of Example 1 occurs often enough that the G_δ-topology not only fails to be Baire, but is actually first category in itself. We will use the following fact:

Lemma 2 (Oxtoby [O]). *Suppose* M *is a separable metric space,* Y *is a regular Baire space, and* A *is a subset of*

$M \times Y$ *such that for each* y *in* Y, *the set* $\{x \in M: (x,y) \in A\}$ *is second category in* M. *Then* A *is second category in* $M \times Y$.

Example 2. Let M be a dense Baire subset of $\mathbb{R}$ such that M has cardinal $\mathbf{M}$. Let $\{x_\alpha: \alpha < \mathbf{M}\}$ be a well-ordering of M. Let $X = \cup_{\alpha<\mathbf{M}}(\{x_\alpha\} \times Y_{\alpha+\omega_0})$, and give X the subspace topology from $M \times Y_{\mathbf{M}}$, so a subset of X is first category in X if and only if it is first category in $M \times Y_{\mathbf{M}}$. Suppose $I \times S$ is a basic open set of $M \times Y_{\mathbf{M}}$, where I is an open interval of M, and let $A = (I \times S) \cap X$. If $y \in S$, say $N_K^\alpha(y) \subseteq S$, then $\{\gamma: (x_\gamma,y) \notin A\} \subseteq [0,\alpha]$; therefore, $\{x: (x,y) \notin A\}$ has cardinality less than $\mathbf{M}$ and so, by the definition of $\mathbf{M}$ is first category in I. Hence, $\{x: (x,y) \in A\}$ is second category in I, so, by Lemma 2, A is second category in $I \times S$. Since A is an arbitrary basic open set, X is Baire. On the other hand, δX is the topological union of the first category spaces $Y_{\alpha+\omega_0}$, $\alpha < \mathbf{M}$, so δX is first category in itself.

References

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