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TWO NORMAL LOCALLY COMPACT SPACES UNDER MARTIN'S AXIOM

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1. Introduction

In [W], S. Watson showed that under V = L, every normal locally compact space is collectionwise T_2 (i.e., every closed discrete collection of points can be separated by a disjoint collection of open sets), and asked if one could go further and get collectionwise normality (i.e., every closed discrete collection of closed sets can be separated). He also asked if in fact the statement

(*) "normal + locally compact + collectionwise T_2 $\Rightarrow$ collectionwise normal"

might be a theorem of ZFC. P. Daniels and the author [DG] answered these questions in the negative by constructing, under V = L, a perfectly normal locally compact collectionwise T_2 non-collectionwise normal space. Now, under MA + $\neg$CH, a perfectly normal counterexample to (*) is impossible: under MA + $\neg$CH, every perfectly normal locally compact collectionwise T_2 space is paracompact (see [G] or [B_1]). However, in this note, we will show that MA + $\neg$CH can be used to construct a normal counterexample. It remains unknown whether or not there is a real counterexample to (*).

In [B_2], Z. Balogh showed that, under $2^{\omega_1} > 2^{\omega}$, every connected locally compact normal Moore space is metrizable

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(or more generally, every connected locally compact normal submetalindelöf space is paracompact). Earlier, Reed and Zenor [RZ] had shown that the above result is a theorem of ZFC if one replaces "connected" by "locally connected" (and Balogh [B_3] has also shown that the corresponding result about covering properties is a theorem of ZFC in this case). Our second example is a rather simple modification of the Cantor tree which, under MA + $\neg$CH, is a connected locally compact normal non-metrizable Moore space, and thus shows that the assumption $2^{\omega_1} > 2^{\omega}$ is necessary in Balogh's theorem.

2. The First Example

Example 1 (MA + $\neg$CH). A normal locally compact collectionwise T_2 non-collectionwise normal space.

In [DG], P. Daniels and the author show that a certain modification of the "Kunen line" construction [JKR] applied to Fleissner's CH example [F] of a normal non-metrizable Moore space gave an example of a perfectly normal locally compact collectionwise T_2 non-collectionwise normal space (in L). Here we show that applying a similar modification of the "van Douwen line" construction [vD] to C. Navy's MA + $\neg$CH example [N] of a normal paralindelöf (hence collectionwise T_2) non-collectionwise normal Moore space yields a space with the properties of Example 1.

First, let us recall Navy's example: under MA + $\neg$CH, there is a normal paralindelöf (hence collectionwise T_2) non-collectionwise normal Moore space Y, having the form

$Y = M \cup J$, where $M = \omega_1{}^{\omega}$ is the product of countably many discrete spaces of size ω_1 and forms a closed metrizable subset of Y, and J is a set of ω_1 isolated points. (For later convenience, we have made here a trivial modification of Navy's space--her space as presented included only increasing functions in $\omega_1{}^{\omega}$, and had 2^{ω} many isolated points.)

Now let us recall the key to van Douwen's construction. This construction generates a finer locally compact locally countable topology on the real line such that, if H and K are disjoint closed sets in the new topology, then $cl_R(H) \cap cl_R(K)$ is countable (this latter fact was the key to proving normality). Here we will construct a finer locally compact locally countable topology τ' on Navy's space (Y,τ) such that, if H and K are closed disjoint in (Y,τ'), then $cl_{\tau}(H) \cap cl_{\tau}(K)$ is σ-discrete. (By "σ-discrete," we mean "σ-closed discrete.") We should point out that van Douwen's construction can be done in ZFC; for our construction, on the one hand we don't need MA + $\neg$CH anymore once we are given Navy's example, but on the other hand we still need $2^{\omega} = 2^{\omega_1}$.

We begin by letting $\{x_{\alpha}: \alpha < \underline{c}\}$ index Y, where $\{x_{\alpha}: \alpha < \omega_1\} = J$, and letting $\{(A_{\alpha},B_{\alpha}): \omega_1 \leq \alpha < \underline{c}\}$ index all pairs (A,B) of subsets of Y of size $\leq \omega_1$ (here's where $2^{\omega} = 2^{\omega_1}$ is used) such that

(i) $|cl_{\tau}(A) \cap cl_{\tau}(B)| = \underline{c}$.

It is not difficult to check, by a simple transfinite induction argument, that one can reindex these sequences to satisfy the following conditions:

(ii) $A_\alpha \cup B_\alpha \subset \{x_\beta\colon \beta < \alpha\}$, and

(iii) $x_\alpha \in cl_\tau(A_\alpha) \cap cl_\tau(B_\alpha)$.

We will assume that our original indexing satisfies these conditions.

We now inductively define a locally compact locally countable topology τ_α on $\{x_\beta\colon \beta < \alpha\}$, finer than the subspace topology in (Y,τ), such that $\beta < \alpha$ implies $\tau_\beta \subset \tau_\alpha$, and such that $x_\alpha \in cl_{\tau_{\alpha+1}}(A_\alpha) \cap cl_{\tau_{\alpha+1}}(B_\alpha)$ for all $\omega_1 \leq \alpha < \underline{c}$. Let τ_α, $\alpha \leq \omega_1$, be the discrete topology. If τ_β has been defined for all $\beta < \alpha$, where $\omega_1 < \alpha < \underline{c}$, we consider two cases:

Case I. *α is a limit ordinal.* In this case, simply let τ_α be the topology generated by $\cup_{\beta<\alpha}\tau_\beta$.

Case II. $\alpha = \gamma + 1$. In this case, $\{x_\beta\colon \beta < \alpha\} = \{x_\beta\colon \beta \leq \gamma\}$, and τ_γ defines a base for all these points except x_γ, so we need to define a base for x_γ. Since $x_\gamma \in cl_\tau(A_\gamma) \cap cl_\tau(B_\gamma)$, we can choose a sequence (y_n) converging to x_γ in τ, with $\{y_n\}_{n\in\omega} \subset A_\gamma \cup B_\gamma \subset \{x_\beta\colon \beta < \gamma\}$, such that both A_γ and B_γ contain infinitely many y_n's. In (Y,τ), there exists a disjoint collection $\{U_n\}_{n\in\omega}$ of open sets with $y_n \in U_n$ such that x_γ is the unique limit point of this collection. Since τ_γ is finer than τ, one can find compact open sets V_n in τ_γ with $y_n \in V_n \subset U_n$. Declare basic neighborhoods of x_γ to have the form

$$B_m(x_\gamma) = \{x_\gamma\} \cup (\cup_{n\geq m} V_n)$$

and let τ_α be the topology on $\{x_\beta\colon \beta \leq \gamma\}$ generated by

$$\tau_\gamma \cup \{B_m(x_\gamma)\colon m \in \omega\}.$$

Let $\tau' = \cup\{\tau_\alpha : \alpha < \underline{c}\}$. We claim that (Y,τ') satisfies the conditions of Example 1. To facilitate the proof, we begin by establishing a few facts.

Fact 1. *If* $H \subset Y$, *then* H *has a dense* σ*-discrete (in* τ*) subset of size* $\leq \omega_1$.

Proof. This is trivial since $Y = M \cup J$, where M is a metric space of weight ω_1, and J is a σ-discrete set of ω_1 isolated points.

Fact 2. *If* H *is closed in* (Y,τ), *and* $|H| < \underline{c}$, *then* H *is* σ*-discrete (in* τ*).*

Suppose H is closed, $|H| < \underline{c}$, but H is not σ-discrete. We can assume $H \subset M$. Then H, being paracompact, cannot be locally σ-discrete--in fact, by first countability, there must be at least two points in H every neighborhood of which is not σ-discrete. Thus there exist disjoint relatively clopen subsets K_0 and K_1 of H, of diameter less than 1, neither of which is σ-discrete. Now there exist non-σ-discrete K_{ij}, $j = 0,1$, of diameter less than 1/2 contained in K_i, $i = 0,1$, and $K_{ijk} \subset K_{ij},\cdots$, so we see that H contains a copy of the Cantor set, which is a contradiction.

The following fact is the one that really makes this construction work.

Fact 3. *If* H *and* K *are disjoint closed subsets of* (Y,τ'), *then*

$$cl_\tau(H) \cap cl_\tau(K) \textit{ is } \sigma\textit{-discrete in } \tau.$$

Proof. Suppose not. Then by Fact 2, $|cl_\tau(H) \cap cl_\tau(K)| = \underline{c}$. Let A and B be dense (in τ) subsets of H and K, respectively, with $|A \cup B| \leq \omega_1$. Then $(A,B) = (A_\alpha, B_\alpha)$ for some α, hence by the construction $x_\alpha \in cl_{\tau'}(A_\alpha) \cap cl_{\tau'}(B_\alpha) \subset H \cap K$, a contradiction.

(Y,τ') *is normal.* Let H and K be disjoint closed sets in (Y,τ'). We will prove that H is contained in the union of countably many open sets whose closures miss K. By symmetry the same will be true of K with respect to H, so H and K can be separated.

By Fact 3, $H \cap cl_\tau(K) = \cup_{n\in\omega} D_n$, where each D_n is closed discrete in τ. Since τ is collectionwise T_2 and normal, there exists a discrete in τ collection $\mathcal{U}_n = \{U_d \colon d \in D_n\}$ separating the points of D_n. For each $d \in D_n$, let V_d be a compact open set in τ' with $d \in V_d \subset U_d$ and $V_d \cap K = \emptyset$. Then $\{V_d \colon d \in D_n\}$ is a discrete collection of clopen sets in τ', so $O_n = \cup\{V_d \colon d \in D_n\}$ is clopen in τ', and $O_n \cap K = \emptyset$. Now let $Y - cl_\tau(K) = \cup_{n\in\omega} W_n$, where each W_n is open and $\overline{W}_n \cap cl_\tau(K) = \emptyset$. (We are using the fact that (Y,τ) is perfectly normal here.) Then $\{O_n \cup W_n\}_{n\in\omega}$ is the desired countable collection of open sets covering H whose closures miss K.

(Y,τ') *is collectionwise* T_2. Let D be a closed discrete subset of (Y,τ'). Let $A \subset D$ be a dense σ-discrete (in τ) subset of D of size $\leq \omega_1$. Since A is closed in τ', from the construction we see that the pair (A,A) never appears in the list $\{(A_\alpha, B_\alpha) \colon \omega_1 \leq \alpha < \underline{c}\}$. It follows that $|cl_\tau(A)| < \underline{c}$, whence $cl_\tau(A)$, and hence D, is σ-discrete in τ. Thus $D = \cup_{n\in\omega} D_n$, where each D_n can be separated (in τ and τ'). By normality, D can be separated in τ'.

(Y,τ') *is not collectionwise normal.* Let $\mathcal{H}$ be a discrete collection of closed subsets of M which cannot be separated in (Y,τ). We will show that $\mathcal{H}$ can't be separated in (Y,τ') either. Suppose on the contrary that $\{U_H: H \in \mathcal{H}\}$ is a disjoint collection of τ'-open sets with $H \subset U_H$. We aim for a contradiction by showing that $\mathcal{H}$ can be covered by a σ-discrete collection $\mathcal{V}$ of τ-open sets such that the closure of each member of $\mathcal{V}$ meets at most one element of $\mathcal{H}$--by a standard substraction argument, this would imply that $\mathcal{H}$ can be separated in τ.

To this end, let $O_H \supset H$ be open in τ such that $cl_\tau(O_H) \cap (\cup\{H' \in \mathcal{H}: H' \neq H\}) = \emptyset$. Without loss of generality, $U_H \subset O_H$. Now H and $Y - U_H$ are disjoint closed subsets of (Y,τ'). By Fact 3, $H \cap cl_\tau(Y - U_H)$ is σ-discrete. Since (Y,τ) is collectionwise T_2, there exists a σ-discrete collection $\mathcal{V}_1$ covering $\{H \cap cl_\tau(Y - U_H): H \in \mathcal{H}\}$, such that each $V \in \mathcal{V}_1$ is τ-open and $\overline{V}$ meets at most one $H \in \mathcal{H}$. Now consider $V_H = Y - cl_\tau(Y - U_H)$. Then $\{V_H: H \in \mathcal{H}\}$ is a disjoint collection of τ open sets covering $(\cup\mathcal{H}) - (\cup\mathcal{V}_1)$. By perfect normality of (Y,τ), we can write $V_H = \cup_{n\in\omega} V_{H,n}$, such that for each n, $\{V_{H,n}: H \in \mathcal{H}\}$ is a discrete collection of τ-open sets. Let $\mathcal{V}_2 = \{V_{H,n}: H \in \mathcal{H}, n \in \omega\}$. Then $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is our desired σ-discrete cover of $\cup\mathcal{H}$. That completes the proof.

3. The Second Example

Example 2. (MA + $\neg$CH). A connected locally compact normal non-metrizable Moore space.

Let T be the Cantor tree, i.e., the n^{th} level of T is the set 2^n of all functions from n into 2, and $t_1 \leq t_2$ if and only if t_2 extends t_1. Let A be a subset of size ω_1 of the Cantor set 2^ω, and let $X = T \cup A$ be the space in which the elements of T are isolated, and a basic neighborhood of a point $x \in A$ is a tail of the sequence $(x|n)_{n\in\omega}$ in T, together with x itself. It is well-known (see, for example, [R]) that X is a locally compact non-metrizable Moore space, and is normal under MA + $\neg$CH.

Now fix some $x_0 \in A$ and "connect up" the sequence $(x_0|n)_{n\in\omega}$ by putting an arc (i.e., a copy of [0,1]) between $x_0|n$ and $x_0|n+1$ for each $n \in \omega$, so that in the resulting space X', we have a connected path from $x_0|0$ to x_0 which contains $x_0|n$ for all n. Clearly X' is still a normal (under MA + $\neg$CH) locally compact Moore space.

Let $Y = (X' \times [0,1]) - (A \times \{1\})$. Now $X' \times [0,1]$ is still a locally compact normal Moore space, and, since it's an open subspace of $X' \times [0,1]$, Y is too. Let $\{t_n: n \in \omega\}$ enumerate the elements of $T - \{x_0|n: n \in \omega\}$. Finally, let Z be the quotient space of Y obtained by identifying $(t_n,1)$ with $(x_0|n,1)$ for each n. We claim that Z satisfies the conditions of Example 2.

Let $f: Y \to Z$ be the quotient map. Since $A \times \{1\}$ is "missing", it is easy to check that f is a closed, hence perfect, map. It follows that Z is a normal locally compact

Moore space, and is non-metrizable since it contains a copy of X. So it remains to prove that Z is connected.

Let S be the arc from $x_0|0$ to x_0 that was added to X to get X'. Notice that the image under f of $(T \cup S) \times [0,1]$ is connected, since we connected every $\{t\} \times [0,1]$, t not in $\{x_0|n: n \in \omega\}$, "at the top" via f to some $\{x_0|n\} \times [0,1] \subset S \times [0,1]$. Thus Z has a dense connected subset, hence is connected.

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