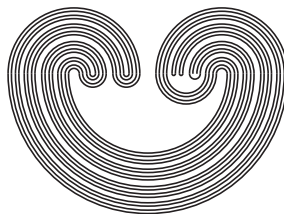

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In [3] Knaster and Reichbach proved that any homeomorphism defined on a closed subset P of the Cantor set C can be extended to a homeomorphism of the Cantor set onto itself. It was proven in [1] and [2] that if P is closed and nowhere dense, then the extended homeomorphism has the following property: an orbit of a point in $C-P$ is dense in C . In fact, there is a dense G_δ of such points. We prove here that in a special case, when the given homeomorphism is an identity on P , an orbit of any point in $C-P$ is dense in $C-P$.

By C we denote the Cantor set in the closed unit interval $[0,1]$ done by the usual ternary construction.

Let β_n denote the family of all sets $[\frac{k-1}{3^n}, \frac{k}{3^n}] \cap C$ that consist of more than two points, where $k = 1, 2, \dots, 3^n$.

Hence β_n is a family of 2^n closed-open and disjoint subsets of C and $\beta = \cup\{\beta_n : n = 1, 2, 3, \dots\}$ is a basis of C .

If f is a homeomorphism on C then, for any integer n , f^n is defined by

$$\begin{aligned}f^0(x) &= x \\f^{n+1}(x) &= f(f^n(x))\end{aligned}$$

where $x \in C$.

Theorem. If D is a closed subset of the Cantor set C then there exists a homeomorphism f from C onto itself such

that $f|_D = \text{id}|_D$ and for every point $c \in (C - D)$ the set $\{f^n(c) : n \text{ is an integer}\}$ is dense in $C - D$.

Proof. If $D = \emptyset$ or $D = C$ then the theorem is true. Suppose then that D is a proper non-empty subset of C . Let \mathcal{U} be a family of pairwise disjoint elements of β such that $\cup \mathcal{U} = C - D$.

For every $B \in \beta$ let $B(1)$ and $B(2)$ denote elements of β such that $\text{diam}B(1) = \text{diam}B(2) = 3^{-1} \cdot \text{diam}B$, $B(1) \cup B(2) = B$, and if $x_1 \in B(1)$ and $x_2 \in B(2)$ then $x_1 < x_2$.

For every positive integer n we define a function f_n and a family \mathcal{A}_n such that the following conditions are fulfilled:

- (i) If $A \in \mathcal{A}_n$ then $\text{diam}A \leq 3^{-n}$.
- (ii) $\cup \mathcal{A}_{n-1} \subseteq \cup \mathcal{A}_n$.
- (iii) If $U \in \mathcal{U}$ and $\text{diam}U \geq 3^{-n}$ then $U \subseteq \cup \mathcal{A}_n$.
- (iv) Let $B \in \beta_n$. If $B \cap D = \emptyset$ then $B \subseteq \cup \mathcal{A}_n$. If $B - (\cup \mathcal{A}_{n-1} \cup D) \neq \emptyset$ then there is $U \in \mathcal{U}$ such that $U \subseteq B - (\cup \mathcal{A}_{n-1} \cup D)$ and $U(1), U(2) \in \mathcal{A}_n$.
- (v) If $A \in \mathcal{A}_n$ then $\mathcal{A}_n = \{f_n^i(A) : i = 1, 2, \dots, m(n)\}$, where $m(n)$ is a number of elements in \mathcal{A}_n .
- (vi) For any $A \in \mathcal{A}_n$ a restriction $f_n|_A$ is an increasing and linear function from A onto $f_n(A)$.
- (vii) $f_n|_{C - \cup \mathcal{A}_n} = \text{id}|_{C - \cup \mathcal{A}_n}$.
- (viii) $f_n(A) = f_{n-1}(A)$ for $A \in \mathcal{A}_{n-2}$.
- (ix) For every $x \in C$ we have $|f_n(x) - f_{n-1}(x)| \leq 3^{1-n}$.
- (x) If $B \in \beta_n$ and $B \cap (\cup \mathcal{A}_n - \cup \mathcal{A}_{n-1}) \neq \emptyset$ then there are $A_1, A_2 \in \mathcal{A}_n$ such that $A_1, A_2 \subseteq B \cap (\cup \mathcal{A}_n - \cup \mathcal{A}_{n-1})$ and $f_n(A_1) = A_2$.

Step 1. The family β_1 consists of two sets, say B_1 and B_2 . For $i = 1, 2$ we put $A_1(i) = \{U(s) : s = 1, 2 \text{ and } U \in \mathcal{U} \text{ and } U \subseteq B_i\}$, if $B_i \cap D = \emptyset$. If $B_i \cap D \neq \emptyset$ and $B_i - D \neq \emptyset$ then we put $A_1(i) = \{U(1), U(2)\}$ where $U \in \mathcal{U}$ and $U \subseteq B_i - D$. We put $A_1 = A_1(1) \cup A_1(2)$. Families $A_1(1)$ and $A_1(2)$ are finite, say $A_1(1) = \{A_1, \dots, A_{m(1)}\}$ and $A_1(2) = \{A_1^*, \dots, A_{m(2)}^*\}$. We define f_1 so that

- $f_1|_{A_j}$ is a linear and increasing function from A_j onto A_{j+1} for $j = 1, \dots, m(1)-1$

- $f_1|_{A_{m(1)}}$ is a linear and increasing function from $A_{m(1)}$ onto A_1^*

- $f_1|_{A_j^*}$ is a linear and increasing function from A_j^* onto A_{j+1}^* for $j = 1, \dots, m(2)$

- $f_1|_{A_{m(2)}^*}$ is a linear and increasing function from $A_{m(2)}^*$ onto A_1

$$- f_1|_{C-UA_1} = \text{id}|_{C-UA_1}.$$

It is easy to see that conditions (i)-(x) are fulfilled.

Step $n+1$. Suppose that we have defined families A_k and functions f_k for $k = 1, 2, \dots, n$. Let the elements of β_{n+1} be denoted by B_j , $j = 1, 2, \dots, 2^{n+1}$, in such a way that $(B_{2^i} \cup B_{2^i-1}) \in \beta_n$ for $i = 1, 2, \dots, 2^n$. Let $A_{n+1} = \{A(s) : s = 1, 2 \text{ and } A \in A_n\}$. Let $A_{n+1}(j) = \{U(s) : s = 1, 2 \text{ and } U \in \mathcal{U} \text{ and } U \subseteq B_j - UA_n\}$ if $B_j \cap D = \emptyset$. If $B_j \cap D \neq \emptyset$ and $B_j - (D \cup UA_n) \neq \emptyset$ then we put $A_{n+1}(j) = \{U(1), U(2)\}$ for some $U \in \mathcal{U}$ such that $U \subseteq B_j - (D \cup UA_n)$.

Let $A_{n+1} = \cup \{A_{n+1}(j) : j = 0, 1, \dots, 2^{n+1}\}$. Conditions (i)-(iv) are satisfied by A_{n+1} .

Let A be a fixed element of A_n . Define function g_0 by

$$- g_0|_{C-A} = f_n|_{C-A}$$

- $g_0|_{A(1)}$ is an increasing and linear function from $A(1)$ onto $f_n(A(2))$

- $g_0|_{A(2)}$ is an increasing and linear function from $A(2)$ onto $f_n(A(1))$.

Conditions (v)-(x) are satisfied by g_0 and $A_{n+1}(0)$.

If $A_{n+1} = A_{n+1}(0)$ then we put $f_n = g_0$.

Suppose that $A_{n+1} \neq A_{n+1}(0)$. Because $(B_{2i-1} \cup B_{2i}) \in \beta_n$ for $i = 1, 2, \dots, 2^n$, then diameter of $\cup(A_{n+1}(2i-1) \cup A_{n+1}(2i))$ is less than or equal to 3^{-n} . Consider $A_{n+1}(1) \cup A_{n+1}(2)$.

If it is an empty set then we put $g_1 = g_0$. Otherwise it is finite. Suppose that both $A_{n+1}(1)$ and $A_{n+1}(2)$ are not empty and put $A_{n+1}(1) = \{A_1, \dots, A_{m(1)}\}$ and $A_{n+1}(2) =$

$\{A_1^*, \dots, A_{m(2)}^*\}$. Because $B_1 \cup B_2$ is an element of β_n and $(B_1 \cup B_2) - (\cup A_{n-1} \cup D) \neq \emptyset$ then, by (iv), (x), and the definitions of $A_n(0)$ and g_0 there are $E_1, E_2 \in A_n(0)$ such that $E_1, E_2 \subseteq (B_1 \cup B_2) - (\cup A_{n-1} \cup D)$ and $g_0(E_1) = E_2$.

We define g_1 from C onto itself as follows:

$$- g_1|_{C-(E_1 \cup \cup A_{n+1}(1) \cup \cup A_{n+1}(2))} =$$

$$g_0|_{C-(E_1 \cup \cup A_{n+1}(1) \cup \cup A_{n+1}(2))}$$

- $g_1|_{E_1}$ is a linear and increasing function from E_1 onto A_1

- $g_1|_{A_r}$ is a linear and increasing function from A_r onto A_{r+1} for $r = 1, 2, \dots, m(1)-1$

- $g_1|_{A_{m(1)}}$ is a linear and increasing function from $A_{m(1)}$ onto A_1^*

- $g_1|_{A_r^*}$ is a linear and increasing function from A_r^* onto A_{r+1}^* for $r = 1, 2, \dots, m(2)-1$

- $g_1|_{A_{m(2)}^*}$ is a linear and increasing function from $A_{m(2)}^*$ onto E_2 .

If one of the families $A_{n+1}(1), A_{n+1}(2)$ is empty then some obvious modifications of the preceding process are required. In any case it is easy to see that the family $A_{n+1}(0) \cup A_{n+1}(1) \cup A_{n+1}(2)$ and the function g_1 satisfy (i)-(x). We repeat this procedure for families $A_{n+1}(2i-1) \cup A_{n+1}(2i)$, where $i = 2, 3, \dots, 2^n$.

If we put $f_{n+1} = g_{2^n}$ then (i)-(x) are fulfilled for such a function and the family A_{n+1} .

Conditions (i)-(x) imply that $f = \lim_{n \rightarrow \infty} f_n$ is a homeomorphism on C that is an identity on D and $\{f^k(c) : k \text{ is an integer}\}$ is dense in $C - D$ for any point c in $C - D$.

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