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Introduction

As a part of the construction of an atriodic, tree-like continuum which is not chainable, the author proved the following:

Theorem [1, Theorem 4, p. 106]. *If* M *is the inverse limit of the inverse limit sequence* $\{X_i, f_i^j\}$ *and there are a positive integer* N *and a positive number* d *such that if* $n \geq N$ *then the span of* f_1^n *is not less than* d, *then* M *has positive span.*

Subsequently, the author obtained a proof of a converse to this theorem and published it [2]. This publication was not widely distributed and consequently many mathematicians did not know of its existence. For example, in a preliminary version of the problems in continuum theory [3] the following problem appeared: If $X = \text{inv lim}\{X_i^{\text{compact}}, f_i^{j\ \text{onto}}\}$ and $\lim_{j\to\infty} \text{span}(f_i^j) = 0$ for a cofinal sequence of i's, is the span of X zero? One purpose of this paper is to provide a positive answer to this question.

1. Preliminaries

Throughout this paper the term *mapping* means continuous function. If X is a space, we denote by p_1 and p_2 the natural projections of $X \times X$ onto X. If Y is a metric space with metric d and f is a mapping of X into Y, by the *span of* f is meant the least upper bound of the set to which the non-negative number c belongs if and only if there is a connected subset Z of $X \times X$ such that $p_1(Z) = p_2(Z)$ and if (x,y) is in Z then $d(f(x),f(y)) \geq c$. By the *span of* X is meant the span of the identity on X. If $X_1, X_2, \cdots$ is a sequence of spaces, denote by p^j the natural projection of $X_1 \times X_2 \times \cdots$ onto X_j. If $X_1, X_2, \cdots$ is a sequence of metric spaces where, for each i, X_i has metric d_i and the diameter of X_i is 1, we consider X metrized by $D(x,y) = \sum_{i=1}^{\infty} 2^{-i} d_i(x_i, y_i)$.

2. Example

At first glance one might suspect that if X is inv $\lim\{X_i, f_i^j\}$ and there is a positive number c such that, for each i, the span of $f_i^{i+1} \geq c$ then X has positive span. That this is not necessarily true may be seen by the following example.

Denote by T the simple triod $OA \cup OB \cup OC$ in the plane given (in polar coordinates) by $T = \{(r,\theta): 0 \leq r \leq 1$ and $\theta = 0, \pi/2, \pi\}$ and $O = (0,0)$, $A = (1,\pi/2)$, $B = (1,\pi)$ and $C = (1,0)$. Denote by f the mapping of T onto T given by:

$$f(r,\theta) = \begin{cases} (r,\pi/2) & \text{if } \theta = \pi/2 \\ (2r,0) & \text{if } \theta = 0 \text{ and } 0 \leq r \leq 1/2 \\ (2-2r,0) & \text{if } \theta = 0 \text{ and } 1/2 \leq r \leq 1 \\ (2r,0) & \text{if } \theta = \pi \text{ and } 0 \leq r \leq 1/4 \\ (1-2r,0) & \text{if } \theta = \pi \text{ and } 1/4 \leq r \leq 1/2 \\ (2r-1,\pi) & \text{if } \theta = \pi \text{ and } 1/2 \leq r \leq 1. \end{cases}$$

It is not difficult to see that the span of f is not less than 1/2. In fact, the set $Z = (\{A\} \times BC) \cup (\{B\} \times AC) \cup (\{C/2\} \times AB) \cup (BC \times \{A\}) \cup (AC \times \{B\}) \cup (AB \times \{C/2\})$ is a subset of $T \times T$ such that $d(f(x),f(y)) \geq 1/2$ for each point (x,y) of Z. We now show that if X is the inv $\lim\{X_i, f_i\}$ where, for each i, X_i is T and f_i is f, the span of X is zero.

First we show that the span of f^2 is 0. There are mappings g and h such that $g\colon T \to [0,1]$, $h\colon [0,1] \to T$ and $f^2 = hg$. Define $g\colon T \to [0,1]$ as follows:

$$g(r,\theta) = \begin{cases} (1/3)r + 2/3 & \text{for } \theta = \pi/2 \\ (-2/3)r + 2/3 & \text{for } \theta = \pi \\ (-2/3)r + 2/3 & \text{for } \theta = 0 \text{ and } 0 \leq r \leq 1/2 \\ (2/3)r & \text{for } \theta = 0 \text{ and } 1/2 \leq r \leq 1. \end{cases}$$

Define $h\colon [0,1] \to T$ as follows:

$$h(t) = \begin{cases} (-6t + 1, \pi) & \text{for } 0 \leq t \leq 1/6 \\ (6t - 1, 0) & \text{for } 1/6 \leq t \leq 1/4 \\ (-6t + 2, 0) & \text{for } 1/4 \leq t \leq 1/3 \\ (6t - 2, 0) & \text{for } 1/3 \leq t \leq 1/2 \\ (-6t + 4, 0) & \text{for } 1/2 \leq t \leq 2/3 \\ (3t - 2, \pi/2) & \text{for } 2/3 \leq t \leq 1. \end{cases}$$

It is not difficult to show that $f^2 = hg$. Because of this factorization it is easy to see that X is homeomorphic to an inverse limit on intervals and is therefore chainable. Since $g\colon T \to [0,1]$ the span of g is zero. That the span of f^2 is 0 follows from the following theorem.

Theorem 1. *If* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *and the span of one of* f *and* g *is zero, then the span of* gf *is* .

Proof. Suppose Z is a subcontinuum of $X \times X$ such that $p_1(Z) = p_2(Z)$. If the span of f is 0 there is a point (x,y) of Z such that $f(x) = f(y)$. Thus, $gf(x) = gf(y)$.

If the span of g is 0, then $Z_1 = (f \times f)(Z)$ is a subcontinuum of $Y \times Y$ such that $p_1(Z_1) = p_2(Z_1)$, so there is a point (x,y) of Z_1 such that $g(x) = g(y)$. Thus there is a point (x',y') of Z such that $gf(x') = gf(y')$.

Remark. It is not possible to have a mapping f of S^1 (the unit circle in the plane) onto S^1 with positive span such that f^2 has span zero since maps of S^1 to S^1 have positive span if and only if they are essential.

3. Main Theorems

In this section we present the main results of this paper. Throughout this section we assume that the factor spaces are compact metric spaces and the bonding maps are surjective.

Theorem 2. *Suppose* $X = \lim\{X_i, f_i^j\}$ *has positive span and* $n(1), n(2), \cdots$ *is an increasing sequence of positive integers. Then there are a positive number* t *and a positive integer* i *such that if* j *is a positive integer not less than* i *then the span of* $f_{n(i)}^{n(j)}$ *is not less than* t.

Proof. Suppose the span of X is $c > 0$ and Z is a continuum in $X \times X$ such that $p_1(Z) = p_2(Z)$ and if (x,y) is in Z then $D(x,y) \geq c$. Assume that for each positive integer i and positive number t there exists a positive integer j such that the span of $f_{n(i)}^{n(j)}$ is less than t. Let i be a positive integer such that $2^{-n(i)} + 2^{-(n(i)+1)} + \cdots \leq c/2$. Using uniform continuity, there is a positive number t such that if p and q are in $X_{n(i)}$ and $d_{n(i)}(p,q) < t$ then for each $k \leq n(i)$, $d_k(f_k^{n(i)}(p), f_k^{n(i)}(q)) < c/2$.

Suppose j is an integer greater than i so that the span of $f_{n(i)}^{n(j)}$ is less than t.

Let $Z_j = (p^{n(j)} \times p^{n(j)})(Z)$. We show that $p_1 Z_j = p_2 Z_j$. If x is in $p_1 Z_j$ then there is a point (z,w) of Z such that $x = z_{n(j)}$. But z is the second coordinate of some point of Z thus x is in $p_2 Z_j$. Similarly, $p_2 Z_j$ is a subset of $p_1 Z_j$.

Since the span of $f_{n(i)}^{n(j)}$ is less than t there is a point (x,y) of Z_j such that $d_{n(i)}(f_{n(i)}^{n(j)}(x), f_{n(i)}^{n(j)}(y)) < t$, and there is a point (z,w) of Z such that $(p^{n(j)} \times p^{n(j)})(z,w) = (x,y)$. Thus, for $k \leq n(i)$, $d_k(z_k, w_k) < c/2$ and $D(z,w) < [c/4 + c/8 + \cdots + c/(2^{n(i)+1})] + [2^{-(n(i)+1)} + 2^{-(n(i)+2)} + \cdots] < c$. This is a contradiction.

Corollary. If X *is the inverse limit of the inverse limit sequence* $\{X_i, f_i^j\}$ *and there is an increasing sequence* $i_1, i_2, \cdots$ *such that if* i *is a term of this sequence then* $\lim_{j\to\infty} \text{span}(f_i^j) = 0$ *then the span of* X *is zero.*

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