TOPOLOGY PROCEEDINGS
Volume 9, 1984
Pages 319–327

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

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All spaces considered are Hausdorff spaces. A topological space X is called *almost 2-fully normal* if the set of the neighborhoods of the diagonal of X is a uniformity. Every paracompact space is almost 2-fully normal and every almost 2-fully normal space is collectionwise normal [2]. Moreover, although Mary Ellen Rudin's Dowker space is almost 2-fully normal [7,8], every weakly Lindelöf almost 2-fully normal space is countably paracompact [15]. M. J. Mansfield has shown that every GO-space is almost 2-fully normal [17]. In [14] it is shown that a locally compact separable normal M-space of D. K. Burke and E. K. van Douwen is almost 2-fully normal. In the same paper a countably compact non-compact Franklin-Rajagopalan space [5] is considered. It is well known that such a space is normal. Answering a question implicitly contained in [14], we show in this note that every countably compact Franklin-Rajagopalan space is almost 2-fully normal.

In the second section of this paper we consider the property of almost 2-full normality in Σ-products. H. H. Corson has proved that a Σ-product of complete separable metric spaces is almost 2-fully normal ([3], compare [12]). In [13] it has been shown by A. P. Kombarov that for a Σ-product Σ of uncountably many nontrivial paracompact p-spaces the following conditions are equivalent:

a) Each factor space is of countable tightness.

b) Σ is collectionwise normal.

c) Σ is normal.

Kombarov's result suggests that Σ-products of paracompact p-spaces of countable tightness are almost 2-fully normal. In this note we verify this conjecture. In particular, Σ-products of metric spaces are almost 2-fully normal, which answers a question of ([10], p. 48).

We call a subset A of a topological space X a *refiner* of a cover $\mathcal{D}$ of X, if A is a subset of some member of $\mathcal{D}$ [14]. We will use the following characterization of almost 2-full normality.

[1,18] A normal topological space X is almost 2-fully normal if and only if for every open cover $\mathcal{D}$ of X there is a locally finite open cover $\mathcal{H}$ of X such that every refiner of $\mathcal{H}$ with at most 2 elements is a refiner of $\mathcal{D}$.

Let n denote an arbitrary cardinal number greater than 1. If one substitutes n for 2 (finitely many for 2) in the given characterization of almost 2-full normality one gets a characterization of the property of almost n-full normality (almost finite full normality) [17,18,14].

1. Countably Compact Franklin-Rajagopalan Spaces

Let μ be an ordinal and let $(A_\alpha)_{\alpha<\mu}$ be a sequence of infinite subsets of the set ω of natural numbers such that

(i) if $\alpha < \beta < \mu$, then $A_\alpha \subset^* A_\beta$ (i.e. $A_\beta \setminus A_\alpha$ is infinite and $A_\alpha \setminus A_\beta$ is finite),

(ii) there is no infinite subset M of ω such that, for each $\alpha < \mu$, $A_\alpha \subset^* M \subset^* \omega$.

On the set $\mu \cup \omega$ (where μ is considered to be disjoint from ω) a topology is defined as follows: Points of ω are isolated. If $0 \leq \beta < \alpha < \mu$ and F is a finite subset of ω, set $U(\alpha,\beta,F) = (\beta,\alpha] \cup (A_\alpha \backslash A_\beta)\backslash F$, and if $\alpha = 0$ and F is a finite subset of ω, set $U(0,\beta,F) = \{0\} \cup (A_0\backslash F)$. For each $\alpha \in \mu$, $U(\alpha,\beta,F)$ is a basic neighborhood of α. In [21] a topological space of this kind is called a countably compact non-compact Franklin-Rajagopalan space. In the following let T be a countably compact non-compact Franklin-Rajagopalan space whose basic neighborhoods are defined in terms of $(A_\alpha)_{\alpha<\mu}$. Let S be a cofinal subset of μ.

Lemma 1 can be proved by straightforward induction on n.

Lemma 1. *Let* $n \in \omega$ *and let* $[\omega]^n = \{C \subset \omega | \text{card}(C) = n\}$. *Let* $\mathcal{E}$ *be an infinite disjoint subfamily of* $[\omega]^n$. *Then there exists an* $\alpha \in S$ *such that the family* $\{E \in \mathcal{E} | E \subset A_\alpha\}$ *is infinite.*

Lemma 2. *Let* $n \in \omega$. *Then there exists* $k \in \omega$ *such that for each* $E \in [\omega\backslash k]^n = \{C \subset \omega\backslash k | \text{card}(C) = n\}$, *the set* $\{\alpha \in S | E \subset A_\alpha\}$ *is cofinal in* μ.

Proof. Assume that the assertion is wrong for some $n \in \omega\backslash\{0\}$. Since the cofinality of μ is uncountable (see e.g. [21]), there is a $\gamma \in S$ and an infinite disjoint subfamily $\mathcal{E}$ of $[\omega]^n$ such that for each $E \in \mathcal{E}$ and for each $\beta \in S$ with $\gamma < \beta$ we have that $E\backslash A_\beta \neq \emptyset$. On the other hand, by Lemma 1 there exists a $\delta \in S$ such that $\{E \in \mathcal{E} | E \subset A_\delta\}$ is infinite. Let $\beta \in S$ such that $\gamma < \beta$ and $\delta < \beta$. Since $A_\delta \subset^* A_\beta$, we have reached a contradiction.

Now we show that T is *almost* n-*fully normal* where $n \in \omega\setminus\{0,1\}$. Our proof is similar to the corresponding proof given in [14].

Let $\mathcal{C}$ be an open cover of T. Without loss of generality we assume that $\mathcal{C} = \{U(x,\beta_x,F_x) | x \in \mu\} \cup \{\{k\} | k \in \omega\}$. Then $x \mapsto \beta_x$ where $x \in \mu$ defines a regressive function on μ. Since the cofinality of μ is uncountable, there exists $\beta < \mu$ such that $\{\gamma \in \mu | \beta_\gamma < \beta\}$ is cofinal in μ (see e.g. [16, p. 153]). Hence there is a cofinal subset S of μ and a finite subset F of ω such that for each $x \in S$, $(\beta,x] \cup ((A_x\setminus A_\beta)\setminus F)$ is a subset of $U(x,\beta_x,F_x)$. By Lemma 2 there exists a $k \in \omega$ such that for each $E \in [\omega\setminus k]^n$, the set $\{\alpha \in S | E \subset A_\alpha\}$ is cofinal in μ. Set $R = (\beta,\mu) \cup (\cup\{A_x\setminus(A_\beta \cup F \cup k) | x \in S \text{ and } x > \beta\})$. Then R is an open set, and since $\mu\setminus R$ is compact, there is a finite subcollection $\mathcal{C}'$ of $\mathcal{C}$ so that $\mu\setminus R \subset \cup\mathcal{C}'$. Let $\mathcal{R} = \mathcal{C}' \cup \{R\} \cup \{\{x\} | x \notin \cup(\mathcal{C}' \cup \{R\})\}$. Then $\mathcal{R}$ is a locally finite open cover of T.

Let $M \subset R$ such that card $(M) \leq n$. There is an $s \in S$ such that $M \cap \mu \subset (\beta,s]$ and $M \cap \omega \subset A_s$. Thus $M \subset U(s,\beta_s,F_s)$. We conclude that every refiner of $\mathcal{R}$ with at most n elements is a refiner of $\mathcal{C}$. Hence T is almost n-fully normal.

Remark 1. Since a separable almost $\aleph_0$-fully normal space is paracompact [1, Prop. 7], T is not almost $\aleph_0$-fully normal. We do not know whether T is almost finitely-fully normal.

2. Σ-Products of Paracompact p-Spaces

Theorem. A Σ-product of paracompact p*-spaces of countable tightness is almost 2-fully normal.*

Remark (December 1984). Our original proof of this theorem was based on results of [11]. In the meantime Y. Yajima published the following result: If Σ is a Σ-product of paracompact Σ-spaces and Σ is of countable tightness, then Σ is collectionwise normal [23]. (Recall that every paracompact p-space is a Σ-space [20].) Revising our paper, we decided to give a variant of our proof that is based on his Lemma 4. We observe that it follows from our proof that a Σ-product of paracompact first-countable Σ-spaces is almost 2-fully normal (compare [23, Corollary 1]).

Proof. Let Σ be a Σ-product of paracompact p-spaces $(X_i)_{i \in I}$ of countable tightness with base point $p \in \Pi\{X_i | i \in I\}$. In order to simplify the notation we will identify in the proof some subspaces of $X_I = \Pi\{X_i | i \in I\}$ and $X_I \times X_I$ that are in fact only homeomorphic. We will have to consider the Σ-product Σ × Σ with base point (p,p) in its Tychonoff product $X_{I\times\{1\}} \times X_{I\times\{2\}}$. For each countable subset B of $I \times \{1\} \cup I \times \{2\}$, $\mathcal{P}_B$ will denote the projection from Σ × Σ onto $X_B = \Pi\{X_i | i \in B\}$. For a countable subset A of I, $\mathcal{Q}_A$ will denote the projection from Σ onto $X_A = \Pi\{X_i | i \in A\}$. The diagonal of Σ will be denoted by Δ.

A Σ-product is of countable tightness, if each finite product of factor spaces is of countable tightness. Since finite products of paracompact p-spaces of countable tightness are of countable tightness, $\Sigma \times \Sigma$ is of countable tightness (see Remark 1 of [13]). Let $\mathcal{D}$ be an open cover of Σ. Set $U = \cup\{C \times C | C \in \mathcal{D}\}$. Since each factor space of $\Sigma \times \Sigma$ is a paracompact Σ-space, by Lemma 4 of [23] there is a σ-locally finite cover $\mathcal{G}$ of $\Sigma \times \Sigma$ satisfying for each $G \in \mathcal{G}$

(i) there exists a countable subset $R(G)$ of $I \times \{1,2\}$ such that $\mathcal{P}_{R(G)}G$ is a cozero-set in $X_{R(G)}$ and $\mathcal{P}^{-1}_{R(G)}\mathcal{P}_{R(G)}G = G$.

(ii) G is disjoint from Δ or $(\Sigma \times \Sigma)\setminus U$.

In the following we assume that $\mathcal{G} = \cup\{\mathcal{G}_n | n \in \omega\}$ where, for each $n \in \omega$, $\mathcal{G}_n$ is locally finite. Let $G \in \mathcal{G}$.

We choose a countable subset $T(G)$ of I such that $R(G) \subset T(G) \times \{1,2\}$. Set $S(G) = T(G) \times \{1,2\}$. Since $X_{T(G)}$ is a countable product of paracompact Σ-spaces, $X_{T(G)}$ is a paracompact Σ-space [20]. Note that $G = \mathcal{P}^{-1}_{S(G)}\mathcal{P}_{S(G)}G$ and that $\mathcal{P}_{S(G)}G$ is a cozero-set in $X_{S(G)}$. Since $X_{T(G)}$ is a paracompact Σ-space, $X_{S(G)} = X_{T(G)} \times X_{T(G)}$ is a rectangular product [22]. Hence $\mathcal{P}_{S(G)}G = \cup\{\cup\mathcal{M}_k(G) | k \in \omega\}$ where for each $k \in \omega$ $\mathcal{M}_k(G)$ is a collection of cozero-set rectangles in $X_{T(G)} \times X_{T(G)}$ that is locally finite in $X_{S(G)}$ [9, Lemma 1].

For each $k \in \omega$ let $\mathcal{N}_k(G) = \{\mathcal{Q}^{-1}_{T(G)}(C \cap D) | C \times D \in \mathcal{M}_k(G)\}$. For each $n,k \in \omega$ set $\mathcal{R}_{nk} = \cup\{\mathcal{N}_k(G) | G \in \mathcal{G}_n\}$. Let $\mathcal{R} = \cup\{\mathcal{R}_{nk} | n,k \in \omega\}$.

We show that $\mathcal{R}$ is a normal open cover of Σ such that $\cup\{K \times K | K \in \mathcal{R}\} \subset U$. Let $\emptyset \neq K \in \mathcal{R}$. Then there are $n,k \in \omega$

such that $K \in \mathcal{R}_{nk}$. Therefore there are $G \in \mathcal{G}_n$ and $C \times D \in \mathcal{M}_k(G)$ such that $K = Q^{-1}_{T(G)}(C \cap D)$. Then $K \times K = \mathcal{P}^{-1}_{S(G)}[(C \cap D) \times (C \cap D)] \subset \mathcal{P}^{-1}_{S(G)}(C \times D) \subset \mathcal{P}^{-1}_{S(G)}\mathcal{P}_{S(G)}G = G \subset U$. We show that $\mathcal{R}$ is a cover of Σ. Let $x \in \Sigma$. Then $(x,x) \in G$ for some $G \in \mathcal{G}$. There is an $n \in \omega$ such that $G \in \mathcal{G}_n$. Hence there are a $k \in \omega$ and a cozero-set rectangle $C \times D \in \mathcal{M}_k(G)$ such that $\mathcal{P}_{S(G)}(x,x) \in C \times D$. Hence $Q_{T(G)}(x) \in C \cap D$ and $x \in \cup\mathcal{R}_{nk}$.

Obviously, each member of $\mathcal{R}$ is a cozero-set of Σ. It remains to show that, for each $n,k \in \omega$, $\mathcal{R}_{nk}$ is locally finite [19, Theorem 1.2]. Let $n,k \in \omega$ and let $x \in \Sigma$. There is an open neighborhood E of x such that $E \times E$ hits only finitely many members of $\mathcal{G}_n$. List these sets as $G_0,\cdots,G_s$. For each $j \in \{0,\cdots,s\}$ there is an open neighborhood M_j of $Q_{T(G_j)}(x)$ in $X_{T(G_j)}$ such that $M_j \times M_j$ hits only finitely many members of $\mathcal{M}_k(G_j)$. Let $M_x = E \cap [\cap^s_{j=0}Q^{-1}_{T(G_j)}M_j]$. Then M_x is an open neighborhood of x in Σ. We show that M_x hits only finitely many members of $\mathcal{R}_{nk}$. Let $y \in M_x \cap K$ with $K \in \mathcal{R}_{nk}$. Then $K = Q^{-1}_{T(G)}(C \cap D)$ where $G \in \mathcal{G}_n$ and $C \times D \in \mathcal{M}_k(G)$. Then $Q_{T(G)}(y) \in C \cap D$. Hence $\mathcal{P}_{S(G)}(y,y) \in C \times D \subset \mathcal{P}_{S(G)}G$ and $(y,y) \in G \cap E \times E$. Therefore $G = G_j$ for some $j \in \{0,\cdots,s\}$. Moreover, for this $j \in \{0,\cdots,s\}$ we have that $Q_{T(G_j)}(y) \in M_j \cap C \cap D$. Only finitely many rectangles in $\mathcal{M}_k(G_j)$ satisfy the last condition. Hence $\mathcal{R}_{nk}$ is locally finite. We conclude that Σ is almost 2-fully normal.

Remark 2. By Lemma 3 of [4] we see that a Σ-product of paracompact p-spaces of countable tightness is in fact almost n-fully normal for every $n \in \omega\setminus\{0,1\}$. Note that we can get this result directly, if we consider Σ^n instead of $\Sigma \times \Sigma$ in the proof given above. In [4] it is shown that a Σ-product of uncountably many copies of the integers is almost $\aleph_0$-fully normal. We do not know whether each Σ-product of paracompact p-spaces of countable tightness is almost $\aleph_0$-fully normal (almost finitely-fully normal).

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