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OBSERVATIONS ON THE PSEUDO-ARC

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At the March 1984 spring topology conference I gave a talk entitled "Survey of the pseudo-arc." It attempted to give a simple, inductive construction of the pseudo-arc as well as give heuristic arguments for some of its basic properties such as homogeneity, uniqueness, and hereditary equivalence. These results are of course well known [1,2,8] but the reasons for them are less well known. After considerable urging from others, consideration is being given to preparing a comprehensive monograph on hereditarily indecomposable continua in order to make this material accessible to a wider community. This is a project which will however require time and must yield precedence to other matters.

In this note, we have a much more limited goal. There are several facts or observations about the pseudo-arc which have in the past proven useful or show prospect of proving useful in the future. Some may turn out only to be interesting curiosities. Some have been known for several years, though apparently not previously appearing in print. Some are more recent observations. No claim is made as to the profundity of any of them, but when I have on occasions referenced them several other persons have seemed surprised, as if they were to them new or unexpected. It therefore seems appropriate to collect

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some of them here. There are undoubtedly others which could be additionally included. Familiarity with the construction of the pseudo-arc and with some essentials about chains and patterns will be assumed.

Close Pseudo-Arcs

One fact which has proven useful in working with continuous decompositions of continua into pseudo-arcs is the fact that any two pseudo-arcs which are close together in terms of Hausdorff distance are also homeomorphically close together. This frequently allows one to construct continuous decompositions into pseudo-arcs with properties which would not be possible for continuous decompositions into less "pathological" continua.

Formally the result is the following.

Theorem 1. *Let* P *be a pseudo-arc embedded in the metric space* X, *and let* $\varepsilon > 0$. *There exists* $\delta > 0$ *such that if* Q *is a pseudo-arc embedded in* X *such that the Hausdorff distance* $H(P,Q) < \delta$ *then there is a homeomorphism* $h: P \twoheadrightarrow Q$ *such that* $\text{dist}(p,h(p)) < \varepsilon$ *for every* $p \in P$.

Proof. Let $C = \{C_0, C_1, \cdots, C_n\}$ be a chain of mesh less than $\varepsilon/3$ which irreducibly covers P and consists of open subsets of X. There exists $\delta > 0$ such that the δ-neighborhood of P is a subset of $C^* = \cup_{i=0}^{n} C_i$ and such that each C_i contains a $p_j \in P$ which is a distance greater than δ from all other C_j's. This δ is the one required by the theorem. Let Q be a pseudo-arc embedded in X such that $H(P,Q) < \delta$. Then $Q \subset C^*$, and for each i,Q contains a point in $C_i - \cup_{\substack{j=0 \\ j\neq i}}^{n} C_j$, since Q contain a point q with $\mathrm{dist}(p_i,q_i) < \delta$.

Thus C is a chain irreducibly covering Q. By a minor variation of Lemma 1.1 of [6] (replacing the requirement that $\hat{P}$ be a subcontinuum of P with $\hat{P}$ being a pseudo-arc), there exists a homeomorphism h: P $\twoheadrightarrow$ Q such that $h(C_i) \subset (\mathrm{st}(C_i,C))^*$ for each link C_i of C. Since mesh $C_i < \varepsilon/3$, we have diam $(\mathrm{st}(C_i,C))^* < \varepsilon$ and $\mathrm{dist}(p,h(p)) < \varepsilon$ for each $p \in P$.

Z-Sets in the Pseudo-Arc

A compactum C in the metric space X is a *Z-set* if for every $\varepsilon > 0$ there exists a map $f: X \to X - C$ such that $\mathrm{dist}(x,f(x)) < \varepsilon$ for each $x \in X$. For the pseudo-arc P it is easy to see that any compactum C not intersecting every composant of P is a Z-set, since in any composant not intersecting C there are arbitrarily large subcontinua of P which are Hausdorff close to P and thus ε-homeomorphic to P. In particular, every proper subcontinuum of the pseudo-arc is a Z-set. However, in addition, nearly every compact subset of the pseudo-arc is a Z-set as is shown by the following.

Theorem 2. If P *is a pseudo-arc then the subspace of* 2^P *(the hyperspace of non-empty subcompacta of* P*) consisting of compacta which are* Z*-sets in* P *is a dense* G_δ *in* 2^P.

Proof. For any compactum X, the subspace of 2^X consisting of Z-sets in X is always a G_δ. Let Z_ε consist of all subcompacta C of X such that there is a map $f: X \to X - C$ which is ε-close to the identity. Then Z_ε is open, since for each C and f any subcompactum of $X - f(X)$ is also in Z_ε. The collection of Z-sets of X is simply $\cap_{n=1}^{\infty} Z_{1/n}$, a G_δ.

For the pseudo-arc the collection of Z-sets is also dense. Let C be a compactum in P and $\varepsilon > 0$. There exists a proper subcontinuum $\hat{P}$ of P and a homeomorphism $h: P \to \hat{P}$ such that $\text{dist}(p,h(p)) < \varepsilon$ for each $p \in P$. Since $\hat{P}$ is a proper subcontinuum of P, every subcompactum of $\hat{P}$ (in particular h(C)) is a Z-set in P. The Hausdorff distance from C to h(C) is less than ε. Since this can be done for every C and every $\varepsilon > 0$, the Z-sets of P are dense in 2^P.

A Z-set must have arbitrarily large subcontinua in its complement. As noted above, any compact subset of the pseudo-arc which fails to intersect every composant satisfies this condition. One might at first be inclined to expect that any compact subset of the pseudo-arc which fails to separate the continuum is a Z-set. However, this is apparently not the case. For a simpler chainable continuum to examine, consider the point (0,1) in the closure of the graph of $f(x) = \sin(1/x)$, $0 < |x| \leq 1$. This point is a nonseparating compact set which is not a Z-set. It should be possible to use this as a guide for an analogous construction with the pseudo-arc. (Merely considering the pre-image of (0,1) under a continuous surjection from the pseudo-arc to the above double $\sin(1/x)$ curve is not necessarily sufficient. The pre-image will fail to be a Z-set, but under some surjections may separate the pseudo-arc.)

Maps onto Weakly Chainable Continua

A continuum is weakly chainable [3,4] if it is the continuous image of a chainable continuum, in particular of the pseudo-arc. Considerable work has been done (so far unsuccessfully) in attempting to prove that every weakly chainable, atriodic, tree-like continuum is chainable. This is equivalent to proving that for every $\varepsilon > 0$ there is a map from such a continuum onto a chainable continuum with each point inverse having diameter less than ε. Though such a proof has so far been elusive, it is fairly easy to construct ε-maps going in the other direction.

Theorem 3. *If* W *is a non-degenerate weakly chainable continuum and* $\varepsilon > 0$, *there exists a map* $f: P \twoheadrightarrow W$, *with* P *a pseudo-arc, such that diam* $(f^{-1}(w)) < \varepsilon$ *for each* $w \in W$.

Proof. Let $g: P \twoheadrightarrow W$ be any continuous surjection. Let C be a chain of mesh less than $\varepsilon/3$ covering the pseudo-arc P. Let n be the number of links in C. There exists a chain D which contains n links and covers W. (Such a chain exists for every nondegenerate continuum though there is in general no control on the size of the links.) $\hat{D}$, the collection of pre-images under g of elements of D, is then also a chain containing n links which covers the pseudo-arc. Since C and $\hat{D}$ contain the same number of links, one can construct a homeomorphism h of the pseudo-arc, such that for each link $\hat{D}(i)$ of $\hat{D}$, $H(\hat{D}(i)) \subset (st(C(i),C))^*$. Then $f = gh^{-1}$ is the desired ε-map from P onto W. If $w \in D(i)$, then $g^{-1}(w) \subset \hat{D}(i)$, and $H(\hat{D}(i)) \subset (st(C(i),C))^*$. So $hg^{-1}(w) = f^{-1}(w)$ has diameter less than ε for each $w \in W$.

Though the pseudo-arc is usually considered as a chainable continuum, there are times when it has proven useful to consider it as an inverse limit of trees or of other finite graphs. The above observation shows that the pseudo-arc is actually M-like for any non-degenerate finite complex M.

Other Observations

The pseudo-arc [5] admits periodic homeomorphisms of every period $n \geq 2$, effective actions by p-adic Cantor groups, and effective actions by every inverse limit of finite solvable groups. Brechner has asked [7] whether it also admits non-periodic, pointwise periodic homeomorphisms. (Any such homeomorphism would have to fail to be regular--i.e. the collection of its iterates would not be equicontinuous.) Such homeomorphisms do exist.

Theorem 4. *The pseudo-arc (and certain other indecomposable chainable continua) admits a non-periodic, pointwise periodic homeomorphism.*

The proof is constructive and in the same spirit as the previous proofs for compact group actions. However it makes use of inverse sequences of finite graphs, rather than trees as in the previous constructions. The details of the construction are tedious and do not seem to offer new tools or insights not inherent in earlier constructions, so we will refrain from presenting them here.

Our final observation has to do with graphics and computer modeling. At the conference there was significant informal discussion among the participants concerning uses they had found for their personal computers. A rumor was circulating to the effect that I had a graphics program that would sketch the pseudo-arc, or at least the first several stages in its construction with ability to increase resolution or expand detail in any desired region. The rumor was a significant exaggeration both of what exists and what is physically possible. It is fairly easy to write a program indicating a pattern for one chain which is crooked in another chain. Beyond that one starts encountering problems. If one's initial chain contains nine links, any crooked chain refining it must have a minimum of 409 links and any pattern for such a chain must contain at least 238 bends. This is within the imaging capacity of many personal computers and monitors. However the minimum number of links in crooked refining chains goes up almost exponentially with the number of links in the initial chain. If one starts with twelve links any crooked refining chain is sufficiently complicated to strain the capacity of more sophisticated professional image processing systems.

The real difficulty comes in if one attempts to indicate further stages in the construction with additional levels of refining chains. If one starts with a chain of nine links, constructs a minimal crooked chain refining it, and then tries to construct a third chain which is crooked and refines this second chain, one will require a minimum of approximately 5.26×10^{155} links, greatly exceeding the classical Jeans estimate for the number of particles in the universe. Even trying to image a small segment of this pattern would appear unmanageable.

What might be possible if one wanted to indicate multiple stages of the construction is to begin with a chain of only five links. This is the first point at which a refining chain can show significant crookedness. One can construct a crooked refining chain with only 13 links. One can then construct a third chain which is crooked in and refines this second chain. At least if one restricts one's attention to subchains of it, it can be imaged without excessive difficulty. Fortunately, it is possible to work with the pseudo-arc without having to have explicit depictions of what happens at successive refining levels.

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