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In [4] H. W. Martin obtains several weak base metrization theorems which generalize well-known theorems belonging to Frink, Nagata, Ceder-Nagata, Morita, Jones, Stone and Arhangel'skii. However, the generalization of the Ceder-Nagata theorem [4, Theorem 2.4] admits an easy proof.

We assume throughout the paper that all topological spaces are T_1 . Also, we let N denote the set of all positive integers.

Let X be a topological space, and for each $x \in X$ let β_x be a collection of subsets of X which is closed under finite intersections such that each element of β_x contains x . The collection $\beta = \{\beta_x: x \in X\}$ is called a weak base for X [1] if the following condition holds: a subset V of X is open if and only if for each $x \in V$, there is some $B_x \in \beta_x$ such that $B_x \subset V$.

The proof of the following result is a generalization of a technique of Hodel [3, Theorem 2.1].

Lemma. Let X be a topological space and let

$\beta = \{\beta_x: x \in X\}$ be a weak base for X such that:

- (i) For each $x \in X$ $\beta_x = \{B_n(x): n = 1, 2, \dots\}$ and $B_{n+1}(x) \subset B_n(x)$.
- (ii) For each $x \in X$ and each $n \in N$ there is an $m = m(n, x) \geq n$ satisfying: $y \in B_m(x)$ implies $B_m(y) \subset B_n(x)$.

Then $\{\text{int } B_n(x) : n = 1, 2, \dots\}$ is a neighbourhood basis of x .

Proof. Let $x \in X$, $n \in \mathbb{N}$ and $g_n(x) = \text{int } B_n(x)$. Let

$$V = \{y \in X : B_{j_y}(y) \subset B_n(x) \text{ for some } j_y\}.$$

It is clear that $x \in V \subset B_n(x)$, and so it remains to show that V is open. Suppose it is not. Then there is an $y \in V$ and a sequence $\{x_k : k = 1, 2, \dots\} \subset X$ such that $x_k \in B_k(y) - V$ for $k = 1, 2, \dots$; consequently, $B_k(x_k) \not\subset B_n(x)$ and there is a $q_k \in B_k(x_k) - B_n(x)$ for $k = 1, 2, \dots$.

Since $y \in V$ there is a $B_j(y) \subset B_n(x)$. For $k = j+1$ we deduce, by (ii), that $B_m(x_m) \subset B_{j+1}(y)$ for some $m \geq j+1$, therefore $q_m \in B_{j+1}(y) \subset B_j(y) \subset B_n(x)$, a contradiction.

Theorem (Martin [4]). A necessary and sufficient condition that a space X be metrizable is that X have a sequence $\{G_n : n = 1, 2, \dots\}$ of closure-preserving covers which satisfy the following two conditions:

(i) Let $B_n(x) = \cap \{G : x \in G \in G_n\}$; then $\beta = \{\beta_x : x \in X\}$ where $\beta_x = \{B_n(x) : n = 1, 2, \dots\}$, is a weak base for X .

(ii) For each $x \in X$ there exists $H_k(x) \in G_k$ such that $x \in H_k(x)$ for all $k \in \mathbb{N}$ and such that if $B_n(x)$ is given, then there exists an $m = m(n, x)$ for which $\text{cl } H_m(x) \subset B_n(x)$.

Proof. It is not restriction to suppose $B_{n+1}(x) \subset B_n(x)$ for each $x \in X$ and each $n \in \mathbb{N}$. Let $g_n(x) = \text{int } B_n(x)$. If $y \in B_n(x)$ we have, by (i), $B_n(y) \subset B_n(x)$. Therefore, condition $y \in g_n(x)$ implies $g_n(y) \subset g_n(x)$ and, by lemma, X is non-archimedeanly quasi-metrizable.

Now let $A = \{A_n : n = 1, 2, \dots\}$ with $A_n = \{(g_n(x), \text{cl } H_n(x)) : x \in X\}$ for each $n \in \mathbb{N}$. If V is a neighbourhood

of x there is, by (ii), a pair n, m such that $x \in g_m(x) \subset \text{cl } H_m(x) \subset B_n(x) \subset V$. Hence, A is a pair-base for X . Also, if $Y \subset X$ we have

$$\text{cl } U\{g_n(x) : x \in Y\} \subset \text{cl } U\{H_n(x) : x \in Y\} = U\{\text{cl } H_n(x) : x \in Y\}$$

since G_n is closure-preserving. Consequently, each A_n is cushioned and X is a stratifiable γ -space. Then, following [2, Theorem 6.1] X is metrizable. The converse is obvious.

References

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