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In [4] H. W. Martin obtains several weak base metrization theorems which generalize well-known theorems belonging to Frink, Nagata, Ceder-Nagata, Morita, Jones, Stone and Arhangel'skiǐ. However, the generalization of the Ceder-Nagata theorem [4, Theorem 2.4] admits an easy proof.

We assume throughout the paper that all topological spaces are T_1. Also, we let N denote the set of all positive integers.

Let X be a topological space, and for each $x \in X$ let $\mathcal{B}_x$ be a collection of subsets of X which is closed under finite intersections such that each element of $\mathcal{B}_x$ contains x. The collection $\mathcal{B} = \{\mathcal{B}_x : x \in X\}$ is called a weak base for X [1] if the following condition holds: a subset V of X is open if and only if for each $x \in V$, there is some $B_x \in \mathcal{B}_x$ such that $B_x \subset V$.

The proof of the following result is a generalization of a technique of Hodel [3, Theorem 2.1].

Lemma. Let X *be a topological space and let* $\mathcal{B} = \{\mathcal{B}_x : x \in X\}$ *be a weak base for* X *such that:*

(i) *For each* $x \in X$ $\mathcal{B}_x = \{B_n(x) : n = 1,2,\cdots\}$ *and* $B_{n+1}(x) \subset B_n(x)$.

(ii) *For each* $x \in X$ *and each* $n \in N$ *there is an* $m = m(n,x) \geq n$ *satisfying:* $y \in B_m(x)$ *implies* $B_m(y) \subset B_n(x)$.

Then $\{int\ B_n(x): n = 1,2,\cdots\}$ *is a neighbourhood basis of* x.

Proof. Let $x \in X$, $n \in N$ and $g_n(x) = \text{int } B_n(x)$. Let

$$V = \{y \in X: B_{j_y}(y) \subset B_n(x) \text{ for some } j_y\}.$$

It is clear that $x \in V \subset B_n(x)$, and so it remains to show that V is open. Suppose it is not. Then there is an $y \in V$ and a sequence $\{x_k: k = 1,2,\cdots\} \subset X$ such that $x_k \in B_k(y) - V$ for $k = 1,2,\cdots$; consequently, $B_k(x_k) \not\subset B_n(x)$ and there is a $q_k \in B_k(x_k) - B_n(x)$ for $k = 1,2,\cdots$.

Since $y \in V$ there is a $B_j(y) \subset B_n(x)$. For $k = j+1$ we deduce, by (ii), that $B_m(x_m) \subset B_{j+1}(y)$ for some $m \geq j+1$, therefore $q_m \in B_{j+1}(y) \subset B_j(y) \subset B_n(x)$, a contradiction.

Theorem (Martin [4]). *A necessary and sufficient condition that a space* X *be metrizable is that* X *have a sequence* $\{G_n: n = 1,2,\cdots\}$ *of closure-preserving covers which satisfy the following two conditions:*

(i) *Let* $B_n(x) = \cap\{G: x \in G \in G_n\}$; *then* $\beta = \{\beta_x: x \in X\}$ *where* $\beta_x = \{B_n(x): n = 1,2,\cdots\}$, *is a weak base for* X.

(ii) *For each* $x \in X$ *there exists* $H_k(x) \in G_k$ *such that* $x \in H_k(x)$ *for all* $k \in N$ *and such that if* $B_n(x)$ *is given, then there exists an* $m = m(n,x)$ *for which* $\text{cl } H_m(x) \subset B_n(x)$.

Proof. It is not restriction to suppose $B_{n+1}(x) \subset B_n(x)$ for each $x \in X$ and each $n \in N$. Let $g_n(x) = \text{int } B_n(x)$. If $y \in B_n(x)$ we have, by (i), $B_n(y) \subset B_n(x)$. Therefore, condition $y \in g_n(x)$ implies $g_n(y) \subset g_n(x)$ and, by lemma, X is non-archimedeanly quasi-metrizable.

Now let $A = \{A_n: n = 1,2,\cdots\}$ with $A_n = \{(g_n(x), \text{cl } H_n(x)): x \in X\}$ for each $n \in N$. If V is a neighbourhood of x there is, by (ii), a pair n,m such that $x \in g_m(x) \subset \text{cl } H_m(x) \subset B_n(x) \subset V$. Hence, A is a pair-base for X. Also, if $Y \subset X$ we have

$$\text{cl } \cup\{g_n(x): x \in Y\} \subset \text{cl } \cup\{H_n(x): x \in Y\} = \cup\{\text{cl } H_n(x): x \in Y\}$$

since G_n is closure-preserving. Consequently, each A_n is cushioned and X is a stratifiable γ-space. Then, following [2, Theorem 6.1] X is metrizable. The converse is obvious.

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