TOPOLOGY PROCEEDINGS

Volume 9, 1984

Pages 359–365

http://topology.auburn.edu/tp/

DOLD THEOREMS IN SHAPE THEORY

by

Harold M. Hastings and Mahendra Jani

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Research Announcement

DOLD THEOREMS IN SHAPE THEORY[1]

Harold M. Hastings[2] and Mahendra Jani[3]

1. Introduction

D. Coram and P. Duvall [5] introduced approximate fibrations by generalizing the lifting property of cell-like maps. S. Mardešić and T. B. Rushing further extended the concept of Hurewicz fibration by generalizing approximate fibrations for compact ANR's to shape fibrations for compacta. This appears to be the appropriate concept of fibrations in the shape theory of compact metric spaces [9,10,12,14,17]. The homotopy theory of fibrations of ANR's extends well beyond their lifting property. In particular, Dold [6] proved that a map of such fibrations over a common, compact, path-connected base is a fibered homotopy equivalence whenever its restriction to a single fiber is a homotopy equivalence. Our purpose is to announce a Dold theorem and related results for shape fibrations. Details and proofs will appear in [9].

Recently H. Kato [12] proved strong versions of Dold's theorems. Kato used the T. A. Chapman's [4] complement theorem to define strong shape. We shall give direct geometric proofs of these theorems using closed model categories [19,7].

---

[1]Based on a talk "Equivalences of shape fibrations" by M. Jani.

[2]Partially supported by NSF grant MCS 8102053.

[3]Partially supported by a research grant from William Paterson College of New Jersey.

2. Preliminaries

We shall use the following categories: TOP, the usual category of topological spaces and continuous maps; CM, the full subcategory of compact metric spaces; PL, the subcategory of finite polyhedra and piecewise linear maps; TOP/B, the usual category of topological spaces and continuous maps over B. For any category C, pro-C shall denote the category of inverse systems over C [1].

To each compact metric space X, we associate the category of finite polyhedra under X, $X \downarrow \mathrm{PL} \to \mathrm{PL}$, following A. Calder and H. M. Hastings [3]. This yields a strong shape functor CM $\to$ pro-PL $\to$ pro-TOP. A map is a strong shape equivalence if it induces an isomorphism in the strong pro-homotopy category Ho(pro-TOP) [7]. The strong shape category is the category of fractions CM (strong shape equivalences)$^{-1}$.

We recall the criteria of S. Mardešić and T. B. Rushing for a shape fibration in terms of its lifting properties.

Definition [16]. A levelwise map $(p_i)\colon \{E_i\} \to \{B_i\}$ is said to have the homotopy lifting property (HLP) if for each n there is an $m \geq n$, such that for any commutative solid-arrow diagram the indicated filler exists.

$$\begin{tikzcd}
X \times 0 \arrow[r, "h"] \arrow[d] & E_m \arrow[r] \arrow[d] & E_n \arrow[d] \\
X \times I \arrow[r, "H"'] \arrow[urr, dashed, "G"] & B_m \arrow[r] & B_n
\end{tikzcd}$$

Theorem (Mardešic and Rushing [16]). *A (continuous) map* p: $E \to B$ *of compact metric spaces is a shape fibration if and only if for each representation of* B *as the inverse limit of a tower* $\{B_i\}$ *of compact* ANR*'s, there is a similar representation* $\{E_i\}$ *of* E*, and a levelwise map* $\{p_i\}: \{E_i\} \to \{B_i\}$ *with* HLP.

In particular, the inverse limit of a sequence of Hurewicz fibrations of compact ANR's is a shape fibration. However, in general, we cannot require that each map $p_i: E_i \to B_i$ be a Hurewicz fibration without violating the requirement of compactness. Thus shape fibrations are natural geometric analog of pro-fibrations.

In [11] M. Jani has proved Dold-theorems for shape fibrations under some additional movability assumptions, using shrinkable open covers.

3. Main Theorems

We now define fibered strong shape equivalence in the strong shape cateogry. All the spaces considered are compact metric spaces.

Definition 1. A map f: $E \to E'$ of spaces over B is called a fibered strong shape equivalence if for each map $C \to B$, the pullback of f over C is a strong shape equivalence (over C).

Theorem 1. *Let* p: $E \to B$ *and* p': $E' \to B$ *be shape fibrations, and let* f: $E \to E'$ *be a map over* B *and a strong shape equivalence. Then* f *is a fibered strong shape equivalence.*

Sketch of proof. By Mardešić and Rushing theorem of section 2, we can assume that p and p' are inverse limits of maps of sequences $\underline{p}$: $\underline{E} \to \underline{B}$ and $\underline{p}'$: $\underline{E}' \to \underline{B}$ respectively, each with the HLP. Without loss of generality, we can also assume that f is the inverse limit of a strong pro-homotopy equivalence $\underline{f}$: $\underline{E} \to \underline{E}'$. We first replace p and p' by fibrations p and p' in the singular model structure on pro-TOP [3,7]. We then use properties of fibrations in pro-TOP [3].

Corollary 1. *A shape fibration which is also a strong shape equivalence is a fibered strong shape equivalence.*

We now describe our versions of Dold's theorems. Let p: E → B and p': E' → B be shape fibrations. Let f: E → E' be a map over B. We shall show that f is a fibered strong shape equivalence under a variety of additional hypotheses.

Theorem 2. *Suppose that* E, E' *and* B *have finite shape dimensions and all spaces and fibers are pointed continua. If the restriction of* f *to* $p^{-1}(*)$ *is a pointed shape equivalence then* f *is a (pointed) strong shape equivalence over* B.

Sketch of proof. Apply the five-lemma to the long-exact sequence of pro-homotopy groups [17] induced by the shape fibrations p and p'.

$$\begin{array}{ccccccccccc}
\to & \text{pro-}\Pi_{n+1}(B) & \to & \text{pro-}\Pi_n(F) & \to & \text{pro-}\Pi_n(E) & \to & \text{pro-}\Pi_n(B) & \to & \text{pro-}\Pi_n(F) & \to \\
 & \| & & \downarrow\cong & & \downarrow f_* & & \| & & \downarrow\cong & \\
\to & \text{pro-}\Pi_{n+1}(B) & \to & \text{pro-}\Pi_n(F') & \to & \text{pro-}\Pi_n(E') & \to & \text{pro-}\Pi_n(B) & \to & \text{pro-}\Pi_n(F') & \to
\end{array}$$

By using the results of [10] and by induction, we prove the following:

Theorem 3. *If* B *admits a finite closed cover* $\{B_i\}$, $i = 1,2,\cdots,N$ *such that for each* i, *the pullback of* f *over* B_i *is a strong shape equivalence then* f *is a fibered strong shape equivalence.*

Combining the theorem 3 with the result [10,9] that the pullbacks of a shape fibration via two strong shape equivalent maps are fibered shape equivalent, we prove that

Theorem 4. *Suppose* B *admits a finite shrinkable closed cover, and for one point* $\{*\}$ *in each strong shape path component of* B, *the pullback of* f *over* $\{*\}$ *is a strong shape equivalence then* f *is a fibered strong shape equivalence.*

Corollary to Theorem 3. *If* B *admits a finite shrinkable closed cover, then every* CE-*shape fibration is a fibered strong shape equivalence.*

Note that there is an inverse sequence of fibrations $\underline{p}\colon \underline{E} \to \underline{B}$ whose 'fiber' is a pro-trivial yet whose inverse limit is not a shape equivalence. However B being an infinite product of spheres, it does not admit finite shrinkable closed cover.

4. Open Problems

Several interesting and harder problems remain open.

(I) Foremost is the classification problem for shape fibrations. A key step along the way involves a glueing lemma, extending Brown and Heath [2].

(II) The 'best' definition of 'principal shape bundle' remains open. The definition should be general enough to include CE maps but restricted enough to permit a relatively straightforward proof of a classification theorem for such maps. See [8] for classification of open principal fibrations.

(III) Finally, the theory of covering maps appears much richer in geometric content than that of Hurewicz fibrations. It should be very appealing to have a comparable theory of shape covering maps.

References

1. M. Artin and B. Mazur, *Etale homotopy*, Lecture Notes in Math. 100, Springer-Verlag, Berlin-Heidelberg-New York (1969).
2. R. Brown and P. Heath, *Coglueing homotopy equivalences*, Math. Zeet. 113 (1970), 313-325.
3. A. Calder and H. M. Hastings, *Realizing strong shape equivalences*, J. Pure Appl. Alg. 20 (1981), 129-156.
4. T. A. Chapman, *On some applications of infinite dimensional topology to the theory of shape*, Fund. Math. 76 (1972), 181-193.
5. D. S. Coram and P. F. Duvall, *Approximate fibrations*, Rocky Mt. J. of Math. 7 (1977), 275-288.
6. A. Dold, *Partitions of unity in the theory of fibrations*, Annals of Math. (2) 78 (1963), 223-255.
7. D. A. Edwards and H. M. Hastings, *Čech and Steenrod homotopy theory with applications to algebraic topology*, Lecture Notes in Math. 542, Springer-Verlag, Berlin-Heidelberg-New York (1976).
8. ________, *Classifications of open principal fibrations*, Trans. AMS, 240 (1978), 213-220.

9. H. M. Hastings and M. Jani, *Equivalences of shape fibrations*, Glasnik Mat. (to appear).
10. M. Jani, *Induced shape fibrations and fiber shape equivalences*, Rocky Mt. J. of Math. 12 (1982), 305-332.
11. ________, *Cell-like shape fibrations which are fiber shape equivalences*, Topology Proceedings 7 (1982), 225-244.
12. H. Kato, *Shape fibrations and fibre shape equivalences I, II*, Tsukuba J. Math. 5 (1982), 223-246.
13. ________, *Fibre shape categories*, Tsukuba J. Math. 5 (1982), 247-265.
14. J. Keesling and S. Mardešić, *A shape fibration with different shapes*, Pacific J. Math. (2) 84 (1979), 319-331.
15. S. Mardešić, *On the Whitehead theorem in shape theory I, II*, Fund. Math. 91 (1976), 51-64.
16. ________ and T. B. Rushing, *Shape fibrations I*, General Topology and Applics. 9 (1978), 193-215.
17. ________, *Shape fibrations II*, Rocky Mt. J. Math. 9 (1979), 283-298.
18. M. Moszynska, *The Whitehead theorem in the theory of shapes*, Fund. Math. 80 (1973), 221-263.
19. D. G. Quillen, *Homotopical algebra*, Lecture Notes in Math. 43, Springer-Verlag, Berlin-Heidelberg-New York (1967).
20. L. Siebenmann, *Infinite simple homotopy theory*, Indag. Math. 32 (1970), 479-495.

Hofstra University

Hempstead, New York 11530

and

William Paterson College

Wayne, New Jersey 07470