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In [CR] Collins and Roscoe proved the following metrization theorem:

Theorem 1. [CR] In order that a T_1 -space be metrizable it is necessary and sufficient that, for each $x \in X$, there is a countable decreasing local neighborhood basis $\{W(i,x) \mid i \in \mathbb{N}\}$ (where \mathbb{N} denotes the set of natural numbers) satisfying

(A) *if $x \in U$ and U is an open set, then there exists a natural number $n = n(x,U)$ and an open set $V = V(x,U)$ containing x such that $x \in W(n,y) \subseteq U$ whenever $y \in V$.*

In this note the Collins-Roscoe theorem is factored so one can topologically see why the result holds.

Definition 1. A countable local neighborhood basis for a space X is a collection $\mathcal{W} = \{W(i,x) \mid i \in \mathbb{N}, x \in X\}$ of not necessarily open sets such that

- (i) For each $i \in \mathbb{N}$ and $x \in X$, $x \in W(i,x)^0$, and
- (ii) If x is in an open set U , then there exists $n = n(x,U) \in \mathbb{N}$ such that $W(n,x) \subseteq U$.

Definition 2. A space X is quasi-developable [Be] if there is a sequence $\mathcal{G} = \{G_1, G_2, \dots\}$ of collections of open sets such that if x is in an open set $U \subseteq X$, then there exists $n(x,U) = n \in \mathbb{N}$ such that $st(x, G_n) \subseteq U$. The sequence \mathcal{G} is a quasi-development for X .

All undefined terms and concepts are as in [E]. All spaces are T_1 -spaces.

Consider the following conditions on a countable local neighborhood bases $\mathcal{W} = \{W(i,x) \mid i \in \mathbb{N}, x \in X\}$ for a space X :

A(1). Given $W(i,x)^\circ$ there is an open set $V(i,x)$ containing x and a natural number $b(i,x) \geq i$ such that if $y \in V(i,x)$, then $x \in W(b(i,x),y)$,

A(2). Given $W(i,x)^\circ$ there is an open set $V(i,x)$ containing x and a natural number $b(i,x)$ such that if $y \in V(i,x)$, then $x \in W(b(i,x),y) \subseteq W(i,x)^\circ$,

A(3). For each $x \in X$ and $i \in \mathbb{N}$, $W(i+1,x) \subseteq W(i,x)$.

The Michael Line [M] satisfies A(1) and A(2) but not A(3). Heath's plane [H] satisfies A(1) and A(3) but not A(2).

It is clear that if the hypothesis of the Collins-Roscoe Theorem is assumed on a countable local neighborhood base then A(2) and A(3) are satisfied. To see that A(1) is satisfied let $W(i,x)^\circ$ be given. Choose $j(i)$ to be the first natural number such that $W(j(i),x)^\circ$ is properly contained in $W(i,x)^\circ$. Let $V(i,x) = V(x,W(j(i),x)^\circ)$ and $b(i,x) = n(x,W(j(i),x)^\circ)$. If $x \in V(i,x)$, then it follows that $x \in W(b(i,x),x)^\circ \subseteq W(b(i,x),x) \subseteq W(j(i),x)^\circ$. Thus $W(b(i,x),x) \subseteq W(i,x)^\circ \subseteq W(i,x)$. Since the local neighborhood base is decreasing it must follow that $b(i,x) \geq i$.

Theorem 2. Let X be a T_1 -space with a countable local neighborhood basis \mathcal{W} . Then

(i) If \mathcal{W} satisfies A(1) and A(3), then closed subsets of X are G_δ -sets,

(ii) If \mathcal{W} satisfies A(2), then X is a quasi-developable space, and

(iii) If \mathcal{W} satisfies A(2) and A(3) X is a collection-wise normal space.

Proof. (i) This follows immediately from Theorem 11 of [CR].

(ii) Let \mathcal{W} be a local neighborhood basis for X that satisfies A(2). Let $G_0 = \{\{x\} \mid \{x\} \text{ is open in } X\}$. Arbitrarily fix $i \in \mathbb{N}$. For each $x \in X$, $W(i,x)^\circ$ induces an open set $V_1(i,x)$ containing x and a natural number $b_1(i,x)$ such that if $y \in V_1(i,x)$ then $x \in W(b_1(i,x),y) \subseteq W(i,x)^\circ$. Notice that $V_1(i,x) \subseteq W(i,x)^\circ$. Choose $m(i,x) \in \mathbb{N}$ such that

$$W(m(i,x),x) \subseteq V_1(i,x) \cap W(b_1(i,x),x)^\circ.$$

Then $W(m(i,x),x)^\circ$ induces an open set $V_2(i,x)$ containing x and a natural number $b_2(i,x)$ such that if $y \in V_2(i,x)$ then

$$x \in W(b_2(i,x),y) \subseteq W(m(i,x),x)^\circ.$$

Let $G(i,j,k,\ell) = \{V_2(i,x) \mid b_1(i,x) = j, m(i,x) = k, b_2(i,x) = \ell\}$, and let $\mathcal{G} = \{G_0\} \cup \{G(i,j,k,\ell) \mid (i,j,k,\ell) \in \mathbb{N}^4\}$.

It follows that \mathcal{G} is a quasi-development for X since if $\{x\}$ is open in X then $\text{st}(x,G_0) \subseteq U$ where U is any open set containing x . If $\{x\}$ is not open in X and $x \in U$ where U is open in X then choose $i \in \mathbb{N}$ such that $W(i,x) \subseteq U$.

Let

$$V_2(i,z) \in G(i,b_1(i,x), m(i,x), b_2(i,x))$$

such that $x \in V_2(i,z)$. Thus $z \in W(b_2(i,z),x)$ and since $b_2(i,z) = b_2(i,x)$, it follows that

$$x \in W(b_2(i,x),x) \subseteq W(m(i,x),x)^\circ \subseteq V_1(i,x).$$

Hence $x \in W(b_1(i,x),z) \subseteq W(i,x)^\circ \subseteq U$. Since

$$V_2(i, z) \subseteq W(b_1(i, x), z) = W(b_1(i, z), z) \subseteq U$$

it follows that

$$\text{st}(x, G(i, b_1(i, x), m(i, x), b_2(i, x))) \subseteq U$$

and X is a quasi-developable space.

(iii) This is noted in the remark following Theorem 11 of [CR] and specifically proved in Theorem 3 of [CRRR].

Clearly imposing conditions A(1), A(2) and A(3) and a local neighborhood bases \mathcal{W} for a T_1 -space X is equivalent to the conditions in the Collins-Roscoe theorem. This represents the factorization of the Collins-Roscoe theorem.

Proof of Theorem 1. Since X has a countable local neighborhood basis satisfying A(1), A(2) and A(3) it is a quasi-developable space having closed sets G_δ -sets and, thus, is developable [Be]. It is also a collection-wise normal T_1 -space and, hence, metrizable [Bi].

This factorization clearly shows the importance of the countable local neighborhood base being decreasing in the Collins-Roscoe Theorem. It also shows that a countable local neighborhood base that satisfies all the conditions of the Collins-Roscoe Theorem except being decreasing also implies a good deal of structure, i.e., quasi-developability, on the space.

References

- [Be] H. R. Bennett, *On quasi-developable spaces*, Gen. Top. Appl. 1 (1971), 253-262.
- [Bi] R. H. Bing, *Metrization of topological spaces*, Canad. J. of Math. 3 (1951), 175-186.
- [CR] P. J. Collins and A. W. Roscoe, *Criteria for metrization*, Proc. Amer. Math. Soc. (to appear).

- [CRRR] _____, G. M. Reed, M. E. Rudin and A. W. Roscoe, *A lattice of conditions on topological spaces*, Proc. Amer. Math. Soc. 94 (1985), 487-496.
- [E] R. Engelking, *General topology*, Polish Scientific Publishers, Warsaw 1977.
- [H] R. W. Heath, *Screenability, pointwise-paracompactness, and metrization of Moore spaces*, Canad. J. Math. 16 (1964), 763-770.
- [M] E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. 69 (1963), 375-376.

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