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## TOPOLOGICAL EMBEDDINGS OF TRIANGULATED MANIFOLDS IN A FOUR-DIMENSIONAL MANIFOLD

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**TOPOLOGICAL EMBEDDINGS OF  
TRIANGULATED MANIFOLDS IN  
A FOUR-DIMENSIONAL MANIFOLD**

**J.C. Cantrell**

There is a long history to the study of locally flat and locally tame embeddings of manifolds. A few references would be [2], [3], [5]. Most previous results were either for embeddings into manifolds of dimension three or of dimension greater than four. Recent breakthroughs by Michael Freedman have led to many exciting results about four-dimensional manifolds [6], [8]. One such result is the theorem that a one- or three-dimensional manifold, topologically embedded in a four-dimensional manifold, is locally flat if its complement is locally 1-connected (1-LC) at each point [8].

This author has previously studied topological embeddings of triangulated manifolds  $M^m$  into a manifold  $N^n$ ,  $n > 4$ ,  $n - m \neq 2$ , in which each open simplex of  $M$  is locally flat in  $N$ , and, in case  $n - m < 3$ , each closed  $(n-2)$ -simplex of  $M$  is locally flat in  $N$  [3]. If the link of each simplex in  $M$  had the homotopy type of a sphere or cell we could conclude that  $M$  was locally flat in  $N$ . In this note we will extend those results to the case  $n = 4$ . There is no need to be concerned about the link condition, for if  $m = 1, 2$ , or  $3$ , the link will automatically be a sphere or cell.

In the case of embedding triangulated simple closed curves or arcs in a four-manifold, with each open simplex locally flat, trivial range results already tell us that the curves or arcs will be locally flat [1]. A little more work is needed in the case of embedding three-manifolds.

*Theorem.* Let  $M$  be a triangulated three-manifold, topologically imbedded in the interior of a four-manifold  $N$  so that open 3-simplexes are locally flat and closed 2-simplexes are locally flat. Then  $M$  is locally flat at each point.

*Proof.* Let  $T$  be a triangulation of  $M$  such that, for each simplex  $\sigma \in T$ ,  $\sigma$  is locally flat in  $N$  at each of its interior points, and that each 2-simplex is locally flat at each of its boundary points.

By definition,  $M$  will be locally flat in  $N$  at each interior point of its 3-simplexes. We will proceed to establish local flatness at points  $x$  that are interior points of simplexes of successively lower dimension.

If  $x$  is an interior point of a 2-simplex  $\sigma$ , we first consider the case where  $\sigma$  lies in the boundary of  $M$ .  $\sigma$  is a face of a 3-simplex  $\tau \in T$ . We let  $D$  be a small 2-cell neighborhood of  $x$  in  $\sigma$  and use local flatness of  $\tau$  at each of its interior points to fatten a neighborhood of  $x$  in  $\tau$  into a 4-cell  $B$  such that  $Bd B = \Sigma$ ,  $\Sigma - D$  is locally flat in  $N$  and  $B \cap \tau$  is an equatorial cross-section of  $B$ . By Theorem 1 of [4],  $N - \Sigma$  is 1-LC at each point of  $\Sigma$ . This implies that  $N - (B \cap \tau)$  is 1-LC at each point, and, hence, that  $N - M$  is 1-LC at  $x$ .

If  $x$  is an interior point of a 2-simplex  $\sigma$  and is an interior point of  $M$ , let  $\tau_1$  and  $\tau_2$  be the two 3-simplexes of  $T$  that have  $\sigma$  as a face. Let  $B_1$  and  $B_2$  be small 3-cells in  $\tau_1$  and  $\tau_2$ , respectively, that share a common 2-cell  $D$  on their boundaries with  $x$  an interior point of  $D$ . Let  $U$  be a neighborhood of  $x$  such that  $U \cap M \subset B_1 \cup B_2$ . By the above paragraph  $N - B_1$  is 1-LC at  $x$ . Thus, small loops near  $x$  and in  $N - (B_1 \cup B_2)$  can be contracted to a point in a small set in  $U$  and missing  $B_1$ . Let  $\ell$  be such a small loop,  $\ell: \text{Bd } I^2 \rightarrow N - (B_1 \cup B_2)$ , and let  $f$  be an extension of  $\ell$  to  $I^2$  such that  $f(I^2)$  has small diameter, misses  $B_1$  and is contained in  $U$ . If  $f(I^2) \cap B_2 \neq \emptyset$ , select a 3-cell  $G$  in  $\text{Int } B_2$  such that  $f(I^2) \cap B_2 \subset G$ .  $M$  locally separates  $N$  into two components  $v_1$  and  $v_2$ . Suppose that  $\ell(\text{Bd } I^2) \subset v_1$ . Let  $A = f^{-1}(\text{cl } v_2)$ , and use Tietze's Extension Theorem to extend  $f|_{A \cap f^{-1}(B_2)}$  to a map  $f': A \rightarrow B_2$ . Redefine  $f$  to be  $f'$  on  $A$ . By using the local flatness of  $G$ , we can push  $f(I^2)$  off of  $G$  into  $v_1$  so that the image of  $I^2$  misses both  $B_1$  and  $B_2$ , and conclude that  $M$  is locally flat at  $x$ .

Now consider the case that  $x$  is an interior point of a 1-simplex  $\sigma$  in  $T$ . Subdivide  $T$  to get  $T'$  such that  $x$  is an interior point of a 1-simplex  $\sigma'$  in  $T'$  and such that  $M$  is locally flat at each point of  $\text{St}(\sigma') - \sigma'$ .

If  $\sigma \in \text{Bd } M$ , the link of  $\sigma'$ ,  $\text{lk } \sigma'$ , is a polygonal arc with ordered vertices  $v_1, v_2, \dots, v_n$ , and  $v_1$  and  $v_n$  in  $\text{Bd } M$ . Let  $\ell: \text{Bd } I^2 \rightarrow N$  be a small loop near  $\sigma'$  and in the complement of  $M$ . Since closed 2-simplexes are locally flat,  $v_1 * \sigma'$  is locally flat, and we can extend  $\ell$  to a

map of  $I^2$  into  $N$  such that  $\ell(I^2)$  has small diameter and misses  $v_1 * \sigma'$ . By using the compactness of  $\ell(I^2)$  and  $v_1 * \sigma'$ , we can find a point  $v_2'$  of the segment  $\langle v_1, v_2 \rangle$  such that  $\langle v_2, v_2' \rangle * \sigma'$  does not intersect  $\ell(I^2)$ . The local flatness of  $\sigma' * \langle v_1, v_2 \rangle - \sigma'$  allows one to move  $\langle v_1, v_2' \rangle * \sigma'$  onto  $\langle v_1, v_2 \rangle * \sigma'$ , thus pushing  $\ell(I^2)$  off  $\langle v_1, v_2 \rangle * \sigma'$ . Continuing in this manner, we are able to construct a homeomorphism  $h$  of  $N$  onto itself that is the identity outside a small neighborhood of  $\text{St}(\sigma')$ , so that  $h\ell(I^2)$  has small diameter and misses  $M$ . This shows that  $N - M$  is 1-LC at  $x$ .

If  $x$  is an interior point of  $M$ , we will subdivide  $T$  to get  $T'$  so that  $x$  is an interior point of a 1-simplex  $\sigma'$  and  $\text{St}(\sigma')$  is in the interior of  $\text{St}(\sigma)$ . Then  $\text{St}(\sigma')$  is locally flat at each point of  $\text{St}(\sigma') - \sigma'$ .

Note that by the above paragraph  $\sigma' * J$  is locally flat for each arc  $J$  in the simple closed curve  $\ell k(\sigma')$ . We let  $\ell: \text{Bd } I^2 \rightarrow N$  be a small loop in the complement of  $M$  and near  $x$ . Let  $\tau$  be any 1-simplex in  $\ell k(\sigma')$  and extend  $\ell$  to a map  $f$  of  $I^2$  into the complement of  $\sigma' * \tau$  and such that  $f(I^2)$  has small diameter. We then modify  $f$ , just as we did in the above case of a point in the interior of a 1-simplex, to get an extension of  $\ell$  that carries  $I^2$  into the complement of  $\text{St}(\sigma')$  and has small diameter. This establishes local 1-connectedness of the complement of  $M$  at interior points of 1-simplexes of  $M$ .

The only possible points of non-local flatness are then the vertices of  $M$ . It has long been known that there are

no isolated points of non-local flatness in dimensions greater than three [7], so that  $M$  is seen to be locally flat at each point. We also note that it is a simple exercise to use the arguments of this paper to establish local 1-connectedness of the complement of  $M$  at its vertices, after having established local 1-connectedness at interior points of higher dimensional simplexes.

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