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A TOPOLOGICAL PROOF OF PAROVIČENKO'S CHARACTERIZATION OF $\beta N - N$

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The following properties of the remainder $P = \beta N - N$ are well-known:

(1) P is a zero-dimensional compact space without isolated points.

(2) Every two disjoint open F_σ-sets in P have disjoint closures (i.e., P is an F-space).

(3) Every non-empty G_δ-set in P has a non-empty interior.

As shown by the four propositions below, these are in fact properties of a larger class of remainders.

Proposition 1. *For each strongly zero-dimensional space* X *the remainder* $\beta X - X$ *is zero-dimensional.*

Proposition 2. *For each* σ*-compact space* X *the remainder* $\beta X - X$ *has no isolated points.*

Proposition 3. ([GH]). *For each locally compact* σ*-compact space* X *the remainder* $\beta X - X$ *is an* F*-space.*

Proposition 4. ([FG]). *For each locally compact realcompact space* X *every non-empty* G_δ*-set in* $\beta X - X$ *has a non-empty interior.*

We shall call a space P a *Parovičenko space*, if P has properties (1)-(3). Since every infinite compact

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[1]This paper has been written while the author was visiting Miami University (Oxford, Ohio).

F-space contains a copy of βN ([GJ], 14N.5), every Parovičenko space has weight $\geq$ c.

In his 1963 paper [P] Parovičenko established the following two theorems:

First Parovičenko Theorem. Every compact space of weight $\leq \aleph_1$ *is a continuous image of* $\beta N - N$.

Second Parovičenko Theorem. The Continuum Hypothesis implies that every Parovičenko space of weight c *is homeomorphic to* $\beta N - N$.

The original proofs were in Boolean algebraic language, and no topological proofs were available until 1980, when Błaszczyk and Szymański presented in [BS] a proof of the First Parovičenko Theorem using the inverse systems technique. Developing their ideas, we shall present here a topological proof of the Second Parovičenko Theorem. Our terminology and notations follow [E].

We start with two characterizations of Parovičenko spaces:

Lemma. For every compact space P *the following conditions are equivalent.*

(i) P *is a Parovičenko space.*

(ii) *For every continuous mapping* f *of* P *onto a compact metrizable space* X *and every pair* F_1, F_2 *of closed subsets of* X *such that* $F_1 \cup F_2 = X$ *there exists an open-and-closed set* $U \subset P$ *such that* $f(U) = F_1$ *and* $f(P - U) = F_2$.

(iii) *For every continuous mapping* f *of* P *onto a compact metrizable space* X *and every continuous mapping* g *of a* *compact metrizable space* Y *onto* X *there exists a continuous mapping* h *of* P *onto* Y *such that* $gh = f$.

Let us precede the proof of our Lemma by brief comments on conditions (ii) and (iii). Condition (iii) first appeared in [N], where it was proved under CH that $\beta N - N$ satisfies (iii) and that (iii) topologically characterizes $\beta N - N$ in the class of all compact spaces of weight $\aleph_1$ (as a matter of fact, there was one more requirement in the characterization of $\beta N - N$ given in [N], viz. that every non-empty compact metric space M is a continuous image of $\beta N - N$, but this follows directly from (iii) by letting $X = \{0\}$ and $Y = M$). Condition (ii) was introduced in [BS], where it was proved that (i) and (ii) are equivalent for every compact space P and that (iii) implies (ii) (in that paper (iii) is misstated: the requirement that $h(P) = Y$ is missing and without this requirement one cannot show in the proof of the implication (iii) $\Rightarrow$ (ii) that $f(U) = F_1$ and $f(P - U) = F_2$).

Proof of the Lemma. It remains to prove that (ii) implies (iii). Since every compact metrizable space Y is a continuous image of the Cantor set $D^{\aleph_0}$ (see [E], Problem 4.5.9(b)), it suffices to prove (iii) under the additional assumption that $Y = D^{\aleph_0} = \Pi_{i=1}^{\infty} D_i$, where $D_i = D = \{0,1\}$ for $i = 1,2,\cdots$. For every finite sequence $i_1, i_2, \cdots, i_k$ of zeros and ones the set $F_{i_1 i_2 \cdots i_k} = \{(i_1, i_2, \cdots, i_k)\} \times \Pi_{i=k+1}^{\infty} D_i$ is open-and-closed in $D^{\aleph_0}$. Applying (ii) we can define inductively open-and-closed subsets $U_{i_1 i_2 \cdots i_k}$ of P such that

$$f(U_{i_1 i_2 \cdots i_k}) = g(F_{i_1 i_2 \cdots i_k}),$$
$$U_{i_1 i_2 \cdots i_{k-1}0} \cap U_{i_1 i_2 \cdots i_{k-1}1} = \emptyset,$$
$$U_{i_1 i_2 \cdots i_{k-1}0} \cup U_{i_1 i_2 \cdots i_{k-1}1} = U_{i_1 i_2 \cdots i_{k-1}},$$
$$U_0 \cap U_1 = \emptyset \text{ and } U_0 \cup U_1 = P.$$

For each $x \in P$ there is exactly one infinite sequence $i_1, i_2, \cdots$ such that $x \in U_{i_1 i_2 \cdots i_k}$ for $k = 1, 2, \cdots$, and the corresponding intersection $\bigcap_{k=1}^{\infty} F_{i_1 i_2 \cdots i_k}$ consists of a single point, so that by letting

$$h(x) = \bigcap_{k=1}^{\infty} F_{i_1 i_2 \cdots i_k} \text{ for } x \in \bigcap_{k=1}^{\infty} U_{i_1 i_2 \cdots i_k}$$

we define a mapping $h\colon P \to D^{\aleph_0}$. Since the sets $F_{i_1}, F_{i_1 i_2}, \cdots$ are closed in the compact space $D^{\aleph_0}$ and form a decreasing sequence, we have $g(\bigcap_{k=1}^{\infty} F_{i_1 i_2 \cdots i_k}) = \bigcap_{k=1}^{\infty} g(F_{i_1 i_2 \cdots i_k})$, so that $f(x) \in \bigcap_{k=1}^{\infty} f(U_{i_1 i_2 \cdots i_k}) = \bigcap_{k=1}^{\infty} g(F_{i_1 i_2 \cdots i_k}) = g(\bigcap_{k=1}^{\infty} F_{i_1 i_2 \cdots i_k}) = g(h(x))$, and thus $gh = f$. One easily checks that

$$h^{-1}(F_{i_1 i_2 \cdots i_k}) = U_{i_1 i_2 \cdots i_k};$$

the family of all the sets $F_{i_1 i_2 \cdots i_k}$ being a base for $D^{\aleph_0}$, this implies that h is continuous and maps P onto $D^{\aleph_0}$.

Proof of the Second Parovičenko Theorem. It suffices to prove that any Parovičenko spaces P, X of weight $\aleph_1$ are homeomorphic. Since X is embeddable in $I^{\aleph_1}$, one can assume that $X = \varprojlim\{X_\alpha, \pi^\alpha_\beta, \alpha < \omega_1\}$, where X_α are compact metrizable spaces, projections $\pi_\alpha\colon X \to X_\alpha$ are mappings onto, and for each limit number $\lambda < \omega_1$ we have $\varprojlim X_\alpha, \pi^\alpha_\beta, \alpha < \lambda\}$ $= X_\lambda$ (see [E], Proposition 2.5.6). Let $\{V_\alpha, \alpha \in A\}$, where

A is the set of all non-limit countable ordinal numbers, be a base for the space P consisting of open-and-closed sets with $V_1 = P$.

Applying transfinite induction we shall define for each $\alpha < \omega_1$ a countable ordinal number $\phi(\alpha) \geq \alpha$ and a continuous mapping f_α of P onto $X_{\phi(\alpha)}$ such that

(1) $\phi(\beta) < \phi(\alpha)$ and $\pi^{\phi(\alpha)}_{\phi(\beta)} f_\alpha = f_\beta$ for $\beta < \alpha$ and

(2) $f_\alpha(V_\alpha) \cap f_\alpha(P - V_\alpha) = \phi$ if $\alpha \in A$.

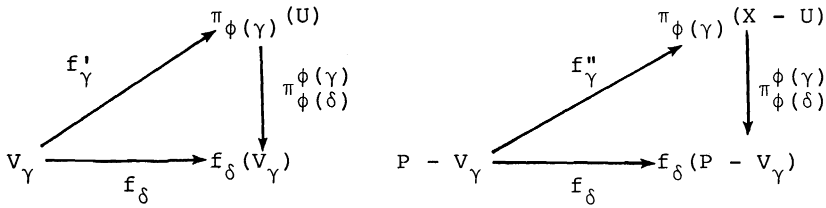
Let $\phi(1) = 1$ and f_1 be an arbitrary continuous mapping of P onto X_1; conditions (1) and (2) are satisfied for $\alpha = 1$. Assume that $\phi(\alpha)$ and f_α are defined for $\alpha < \gamma$ and satisfy conditions (1) and (2).

If γ is a limit number, we define $\phi(\gamma) = \sup\{\phi(\alpha): \alpha < \gamma\}$ and $f_\gamma = \varprojlim\{f_\alpha, \alpha < \gamma\}$. Condition (1) is satisfied for $\alpha = \gamma$ and, by Corollary 3.1.16 in [E], f_γ maps P onto $\varprojlim\{X_{\phi(\alpha)}: \alpha < \gamma\} = X_{\phi(\gamma)}$.

If $\gamma = \delta + 1$, the ordinal number $\phi(\delta)$ and the mapping f_δ are already defined, and the sets $f_\delta(V_\gamma)$, $f_\delta(P - V_\gamma)$ are closed and cover the space $X_{\phi(\delta)}$. Since X is a Parovičenko space, there exists by (ii) an open-and-closed set $U \subset X$ such that

$$\pi_{\phi(\delta)}(U) = f_\delta(V_\gamma) \text{ and } \pi_{\phi(\delta)}(X - U) = f_\delta(P - V_\gamma).$$

The limit of an inverse system of non-empty compact spaces being non-empty, there exists a countable ordinal number $\phi(\gamma)$, larger than both γ and $\phi(\delta)$, such that $\pi_{\phi(\gamma)}(U) \cap \pi_{\phi(\gamma)}(X - U) = \emptyset$. Now, since every non-empty open-and-closed subspace of a Parovičenko space is a Parovičenko space, there exist by (iii) continuous mappings



f'_γ of V_γ onto $\pi_{\phi(\gamma)}(U)$ and f''_γ of $P-V_\gamma$ onto $\pi_{\phi(\gamma)}(X - U)$ such that

$$\pi^{\phi(\gamma)}_{\phi(\delta)} f'_\gamma(x) = f_\delta(x) \text{ for } x \in V_\gamma \text{ and}$$

$$\pi^{\phi(\gamma)}_{\phi(\delta)} f''_\gamma(x) = f_\delta(x) \text{ for } x \in P - V_\gamma.$$

By letting

$$f_\gamma(x) = \begin{cases} f'_\gamma(x), & \text{if } x \in V_\gamma \\ f''_\gamma(x), & \text{if } x \in P - V_\gamma \end{cases}$$

we define a continuous mapping f_γ of P onto $X_{\phi(\gamma)}$ such that (1) and (2) are satisfied for $\alpha = \gamma$.

The limit mapping $f = \varprojlim\{f_\alpha, \alpha < \omega_1\}$ maps P onto $\varprojlim\{X_{\phi(\alpha)}, \alpha < \omega_1\} = X$. To show that f is a homeomorphism, it suffices to observe that by (2) f is a one-to-one mapping.

Let us add that a proof of the First Parovičenko Theorem, in principle identical with the one given in [BS], can be obtained by obvious simplifications in the above proof. It should be also added that, as established in [DM], the assumption that every Parovičenko space of weight c is homeomorphic to the remainder $\beta N - N$ implies the Continuum Hypothesis.

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