TOPOLOGY PROCEEDINGS

Volume 10, 1985

Pages 55–57

http://topology.auburn.edu/tp/

BURKE'S THEOREM FROM PRODUCT CATEGORY EXTENSION AXIOM

by

William G. Fleissner

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

BURKE'S THEOREM FROM PRODUCT CATEGORY EXTENSION AXIOM

William G. Fleissner[1]

Recently, Burke [B] assumed the Product Measure Extension Axiom and proved several statements, including "countably paracompact Moore spaces are metrizable." Soon afterwards, Tall [T] showed that these statements hold in models constructed by adding supercompact many Cohen reals. In this note we show that Burke's statements follow from the Product Category Extension Axiom (PCEA). PCEA holds in models constructed by adding strongly compact many Cohen reals [F]. The combinatorics which allow us to do measure-like arguments in Reg$\mathcal{O}$pen (N^I) are due to Dow [D].

We consider N to be the countable discrete space and N^I to have the Tychonoff, finite support product topology. Basic open sets have the form $[\eta]_I = \{f \in N^I: \eta \subset f\}$, where η is a function from a finite subset of I to N. We denote the cardinality of a set S by $|S|$. Thus $[\eta]$ cuts on $|\eta|$ coordinates. The regular open subsets of a space X form a complete Boolean algebra Reg$\mathcal{O}$pen (X).

PCEA is the assertion that for every index set I, there is $J \supset I$ and a $\mathfrak{c}$-complete Boolean homomorphism $H: P(N^I) \to$ Reg$\mathcal{O}$pen (N^J) such that $h([\eta]_I) = [\eta]_J$ for all basic open sets $[\eta]_I$.

---

[1]Partially supported by NSF grant DMS-8401003.

Theorem 1 [D]. *For every index set* I *there is a nested decreasing sequence* $(L_n)_{n\in\omega}$ *of subsets of the regular open algebra of* N^I *satisfying for each* n

a) *if* $\sup\{b_j: j \in \omega\} = \underline{1}$, *then there is* $k \in \omega$ *such that* $\sup\{b_j: j \leq k\} \in L_n$

b) *if* $b_1, \cdots, b_n \in L_n$ *and* $|\eta| = n$, *then* $b_1 \wedge \cdots \wedge b_n \wedge [\eta] \neq \underline{0}$.

If we assume PCEA, we can pull back Dow's Lemma to $P(N^I)$. For $A \in P(N^I)$, we will write $A \in L_n^*$ if $h(A) \in L_n$.

We will present one version of Burke's result below, and leave the versions with countable paracompactness and/or character below $\mathfrak{c}$ to the reader.

Theorem 2. *Assume* PCEA. *Let* $\mathcal{Y} = \{Y_i: i \in I\}$ *be a discrete collection of closed sets in a countably metacompact first countable space* X. *There is a family of collections of open sets* $\mathcal{H} = \{\mathcal{H}_n: n \in \omega\}$, *where* $\mathcal{H}_n = \{H_i^n: i \in I\}$, *satisfying* α) $Y_i \subset H_i^n$ *for all* n *and* i, *and* β) *for all* $x \in X$ *there is* $n \in \omega$ *such that* $\{i \in I: x \in H_i^n\}$ *is finite.*

Proof. Let $Z = X - \cup\mathcal{Y}$.

For each $f \in N^I$, consider the countable open cover $\mathcal{U}_f = \{Z \cup \cup\{Y_i: f(i) = m\}: m \in \omega\}$. Apply countable metacompactness get a point finite refinement of $\mathcal{U}_f$, $\mathcal{V}_f = \{V_{mf}: m \in \omega\}$. If $y \in Y_i$ and $f(i) = m$, let $V_{yf} = V_{mf}$. For $y \in \cup\mathcal{Y}$, let $\{B_{yj}: j \in \omega\}$ be a base for y. For every $f \in N^I$, there is $j \in \omega$ so that $B_{yj} \subset V_{yf}$; i.e. $\cup\left\{\{f \in N^I: B_{yj} \subset V_{yf}\}: j \in \omega\right\} = P(N^I)$. Let k_{yn} be given by 1 a). Set $W_{yn} = \cap\{B_{yj}: j \leq k_{yn}\}$; this gives, for all $y \in \cup Y$ and $n \in \omega$,

(*) $\{f \in N^I: W_{yn} \subset V_{yf}\} \in L_n^*$.

Set $H_i^n = \cup\{W_{yn}: y \in Y_i\}$. Clearly H_i^n is an open set containing Y_i; we must verify β).

Aiming for a contradiction, let $x \in X$ be such that for all $n \in \omega$, $\{i: x \in H_i^n\}$ is infinite. Set $A_k = \left\{f: |\{V \in \mathcal{V}_f: x \in V\}| = k\right\}$. Because $\cup\{A_k: k \in \omega\} = N^I$ and h is countably complete, there is some $k \in \omega$ and some basic open set $[\eta]$, $|\eta| = j$ such that

(**) $h([\eta]) \leq h(A_k)$.

Let $n = j + k + 1$. Because $J = \{i: x \in H_i^n\} - \text{dom } \eta$ is infinite, there is a one-to-one function θ with $\text{dom } \theta \in [J]^{k+1}$ and range $\theta \subset N$. Now $\eta \cup \theta$ is a function, $|\eta \cup \theta| = j + k + 1$ so by 1b), (*), and (**), there is $f \in [\eta \cup \theta] \cap A_k$ such that for all $y \in \text{dom } \theta$, $W_{yn} \subset V_{yf}$. However, by the definition of H_n and the choice of θ, x is in at least k + 1 V_{mf}'s. This contradiction establishes Theorem 2.

Bibliography

[B] D. K. Burke, PMEA *and first countable, countably paracompact spaces*, Proc. AMS 92 (1984), 455-460.

[D] A. Dow, *Remote points in large products*, Topl. & Appl. 16 (1983), 11-17.

[F] W. G. Fleissner, *Homomorphism axioms and lynxes*, Contemp. Math. 31 (1984), 79-97.

[T] F. D. Tall, *Countable paracompact Moore spaces are metrizable in the Cohen model*, Top. Proc. 9 (1984), 145-148.

University of Pittsburgh

Pittsburgh, Pennsylvania 15260