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MULTICOHERENCE AND PRODUCTS

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MULTICOHERENCE AND PRODUCTS

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Introduction

Let W be any space, we define $b_o(W)$ = (number of components of W) - 1 if this number is finite and $b_o(W) = \infty$ otherwise. W is *insular* when $b_o(W)$ is finite. Let Z be a connected space; the *multicoherence degree*, $r(Z)$, of Z is defined by $r(Z) = \sup\{b_o(A \cap B)$: A, B are closed connected subsets of Z and $Z = A \cup B\}$. Z is said to be *unicoherent* if $r(Z) = 0$.

A *region* of Z is an open connected subset of Z. A *map* is a continuous function. We will denote by $\mathbb{R}$ the real line; by S the unit circle in the complex plane; by S^W the group of maps of W in S with the complex multiplication and by $e: \mathbb{R} \to S$ the exponential map. For $f \in S^W$, we write $f \sim 1$ if there exists a map $g: W \to \mathbb{R}$ such that $e \circ g = f$ and we write $f \not\sim 1$ if this is not true. If $A \subset W$, the restriction of f to A will be denoted by $f|A$.

For two closed subsets A, B of Z, we denote by P(A,B) the subgroup of S^Z which consists of all $f \in S^Z$ such that $f|A \sim 1$ and $f|B \sim 1$. And we define $\mathcal{P}(A,B)$ = maximum number of linearly independent elements of P(A,B) if this number is finite and $\mathcal{P}(A,B) = \infty$ otherwise. A finite number of elements $f_1, \cdots, f_n$ of S^Z is said to be *linearly independent* provided that $f_1^{a_1} \cdot \cdots \cdot f_n^{a_n} \sim 1$ where $a_1, \cdots, a_n$ are integers is possible only when $a_1 = \cdots = a_n = 0$. Finally, we define the *analytic multicoherence degree*, $\mathcal{P}(Z)$, of Z by

$\mathcal{P}(Z) = \sup\{\mathcal{P}(A,B)$: A, B are closed subsets of Z and $Z = A \cup B\}$.

C. Kuratowsky (Fund. Math. 15 (1930) page 353) asked the following: Is the product of two unicoherent Peano continua unicoherent? In [1], K. Borsuk gave an affirmative answer to this question. S. Eilenberg [2] proved that the product of two connected, locally connected, unicoherent metric spaces is unicoherent. And T. Ganea [4] generalized these results proving that the product of an arbitrary family of connected, locally connected, unicoherent spaces is unicoherent.

In [3], S. Eilenberg proved that if X, Y are connected, compact, metric spaces, then $\mathcal{P}(X \times Y) = \sup\{\mathcal{P}(X), \mathcal{P}(Y)\}$. This equality was generalized to denumerable products by A. H. Stone [9] and he mentioned that is valid for arbitrary products of connected, compact, metric spaces. On the other hand, the equality $\mathcal{R}(Z) = \mathcal{P}(Z)$ is proved by S. Eilenberg [3] when Z is a connected, locally connected, compact, metric space. In [8], A. H. Stone showed that this equality holds for all connected, locally connected, normal T_1-spaces.

Using these results, we have that the equality:

$$\mathcal{R}(\Pi X_\alpha) = \sup\{\mathcal{R}(X_\alpha): \alpha \in J\} \qquad (1)$$

holds if each space X_α is a connected, locally connected, compact, metric space. In this paper, we prove that the equality (1) is true if each X_α is a connected, locally pathwise connected, normal T_1-space.

It is important to observe that, while normality is a standard assumption in this area, a product of normal spaces need not be normal. This difficulty is handled by using regularity (instead of normality) for most of the arguments.

1. Some Auxiliary Results

Throughout this section X and Y will denote connected, locally connected, regular spaces. If W is any space, we define $\mathcal{C}(W) = \{D: D \text{ is a component of } W\}$. If m is a positive integer, we define $\overline{m} = \{1,2,\cdots,m\}$. We use the notation $f \approx g$ to indicate that the maps f and g are homotopic.

1.1 *Lemma. Let* B *be a closed subset of* X *and let* U *be an open insular subset of* X *such that* $b_o(Cl_X(U) \cap B) \geq m$. *Then there exists an open subset* V *of* X *such that* $U \subset V$, $b_o(Cl_X(V) \cap B) \geq m$, $b_o(Cl_X(V)) = b_o(V)$ *and* $b_o(U) \geq b_o(V)$.

Proof. Suppose that $b_o(U) - b_o(Cl_X(U)) > 0$, then there exists a point $p \in X$ and there exist two components of U such that each one of them has p in its closure. Since $b_o(Cl_X(U) \cap B) \geq m$, there exist nonempty, pairwise disjoint closed subsets $B_1,\cdots,B_{m+1}$ of X such that $Cl_X(U) \cap B = B_1 \cup \cdots \cup B_{m+1}$. Since X is regular, we can take a region W of X such that $p \in W$ and $Cl_X(W)$ intersects at most one of the B_i's. Then $V_1 = U \cup W$ is an open subset of X such that $U \subset V_1$, $b_o(Cl_X(V_1) \cap B) \geq m$ and $b_o(V_1) < b_o(U)$. Then V can be constructed repeating this argument when necessary.

1.2 *Lemma. Let* B *be a closed connected subset of* X *and let* U *be an open insular subset of* X *such that* $X = B \cup U$ *and* $b_o(B \cap Cl_X(U)) \geq m \geq 1$. *Then there exists a region* W *of* X *such that* $U \subset W$ *and* $b_o(B \cap Cl_X(W)) \geq m - b_o(Cl_X(U))$.

Proof. We can suppose that $m - b_o(Cl_X(U)) > 0$. It will be enough to prove that if U is not connected, then there exists an open subset W_1 of X such that $b_o(W_1) < b_o(U)$, $U \subset W_1$ and $b_o(B \cap Cl_X(W_1)) \geq m - 1$. Suppose then that U is not connected. We take V as in Lemma 1.1. If $Cl_X(V)$ is connected, then V is so. In this case we put $W_1 = V$. Suppose then that $Cl_X(V)$ is not connected. We put $Cl_X(V) = H \cup K$ where H, K are closed, nonempty disjoint subsets of X.

Let $C_1, \cdots, C_s$ be closed, nonempty pairwise disjoint subsets of X such that $s \geq m + 1$, $B \cap Cl_X(V) = C_1 \cup \cdots \cup C_s$ and each C_i is contained in some component of $Cl_X(V)$. We make $I = \{i \in \bar{s}: C_i \subset H\}$, then I and $\bar{s} - I$ are nonempty. It is easy to prove that there exist $i \in I$, $j \in \bar{s} - I$ and D a component of $X - Cl_X(V)$ such that $Cl_X(D) \cap C_i \neq \phi$ and $Cl_X(D) \cap C_j \neq \phi$. We choose points $p \in Cl_X(D) \cap C_i$ and $q \in Cl_X(D) \cap C_j$. Let U_1, U_2 be regions of X such that $p \in U_1$, $q \in U_2$, $Cl_X(U_1) \cap (\cup\{C_k: k \neq i\}) = \phi$ and $Cl_X(U_2) \cap (\cup\{C_k: k \neq j\}) = \phi$. Since $D \cap U_1 \neq \phi$ and $D \cap U_2 \neq \phi$, we have that there exists a region E of X such that $Cl_X(E) \subset D$, $E \cap U_1 \neq \phi$ and $E \cap U_2 \neq \phi$. We define $W_1 = V \cup U_1 \cup U_2 \cup E$.

1.3 *Theorem. If* $\kappa(X) \geq m \geq 1$, *then there exist regions* U, V *of* X *and there exists* $\mathcal{C} \subset \mathcal{C}(X - Cl_X(U))$ *such that* $b_o(H \cap K) \geq m$ *and* $X = H \cup K$ *where* $H = Cl_X(U) \cup (\cup\{D: D \in \mathcal{C}\})$ *and* $K = Cl_X(V)$.

Proof. Let A, B be closed connected subsets of X such that $X = A \cup B$ and $b_o(A \cap B) \geq m$. It is enough to prove that there exist a region U of X and $\mathcal{D} \subset \mathcal{C}(X - B)$ such that

$X = Cl_X(U) \cup (B \cup (\cup\{D: D \in \mathcal{D}\}))$ and $b_o(Cl_X(U) \cap (B \cup (\cup\{D: D \in \mathcal{D}\}))) \geq m$. Let $C_1, \cdots, C_{m+1}$ be closed, nonempty, pairwise disjoint subsets of X such that $A \cap B = C_1 \cup \cdots \cup C_{m+1}$. For $J \subset \overline{m+1}$, we define $C_J = \cup\{C_j: j \in J\}$. We make $\mathcal{J} = \{J \subset \overline{m+1}: J \neq \phi \text{ and } J \neq \overline{m+1}\}$. Then if $J \in \mathcal{J}$, we can choose $D_J \in \mathcal{C}(X - B)$ such that $Cl_X(D_J) \cap C_J \neq \phi$ and $Cl_X(D_J) \cap C_{\overline{m+1}-J} \neq \phi$. We put $\mathcal{D} = \mathcal{C}(X - B) - \{D_J: J \in \mathcal{J}\}$, $U_1 = \cup\{D_J: J \in \mathcal{J}\}$ and $B_1 = B \cup (\cup\{D: D \in \mathcal{D}\}$.

It is not difficult to prove that $b_o(B_1 \cap Cl_X(U_1)) \geq m + b_o(Cl_X(U_1))$. Then, by Lemma 1.2, there exists a region U of X such that $U_1 \subset U$ and $b_o(B_1 \cap Cl_X(U)) \geq m$. This completes the proof.

We denote by $\mathbb{R}^2$ the Euclidean plane. For a positive integer n, we define $\mathcal{L}_n = \{(u,v) \in \mathbb{R}^2: (u - (2i-1))^2 + v^2 = 1 \text{ for some } i \in \overline{n}\}$ ($\mathcal{L}_n$ is a row of n unit circles each touching the next one in a single point), $\mathcal{L}_n^+ = \{(u,v) \in \mathcal{L}_n: v \geq 0\}$ and $\mathcal{L}_n^- = \{(u,v) \in \mathcal{L}_n: v \leq 0\}$, we consider these spaces with the topology that they have as subspaces of $\mathbb{R}^2$. For $i \in \overline{n}$, we define $\ell_i: \mathcal{L}_n \to S$ by:

$$\ell_i(u,v) = \begin{cases} (u - (2i-1), v) & \text{if } |u - (2i-1)| \leq 1 \\ (-1,0) & \text{if } u \leq 2i - 2 \\ (1,0) & \text{if } u \geq 2i \end{cases}$$

(ℓ_i is "essentially" the retraction of $\mathcal{L}_n$ in its i^{th} circle)

1.4 *Proposition. Suppose that* X *is locally pathwise connected and that* $r(X) \geq m \geq 1$. *Then there exist closed connected subsets* A, B *of* X; *there exist closed, nonempty, pairwise disjoint subsets* $C_1, \cdots, C_{m+1}$ *of* X *and there exists* *a map* $\sigma\colon \mathcal{L}_m \to X$ *such that* $\sigma(\mathcal{L}_m^+) \subset A$, $\sigma(\mathcal{L}_m^-) \subset B$, $A \cap B = C_1 \cup \cdots \cup C_{m+1}$ *and* $\sigma(2i-2,0) \in C_i$ *for each* $i \in \overline{m+1}$.

Proof. Let U, V, $\mathcal{C}$, H and K as in Theorem 1.3. Suppose that $H \cap K = E_1 \cup \cdots \cup E_{m+1}$ where $E_1,\cdots,E_{m+1}$ are closed, nonempty, pairwise disjoint subsets of X. From the connectedness of V it follows that $E_i \cap Cl_X(U) \cap Cl_X(V) \neq \phi$ for each $i \in \overline{m+1}$. We choose points $p_1 \in E_1 \cap Cl_X(U)$, $\cdots,p_{m+1} \in E_{m+1} \cap Cl_X(U)$ and we take regions $U_1,\cdots,U_{m+1}$ of X such that $p_1 \in U_1,\cdots,p_{m+1} \in U_{m+1}$ and $Cl_X(U_i) \cap (\cup\{E_j \cup Cl_X(U_j)\colon j \neq i\}) = \phi$. We define $U_0 = U \cup U_1 \cup \cdots \cup U_{m+1}$, $V_0 = V \cup U_1 \cup \cdots \cup U_{m+1}$, $A = H \cup Cl_X(U_0)$, $B = K \cup Cl_X(V_0)$ and $C_1 = E_1 \cup Cl_X(U_1),\cdots,C_{m+1} = E_{m+1} \cup Cl_X(U_{m+1})$. Since U_0 and V_0 are regions of X, there exist maps $\sigma_1\colon \mathcal{L}_m^+ \to U_0$ and $\sigma_2\colon \mathcal{L}_m^- \to V_0$ such that $\sigma_1(2i-2,0) = p_i = \sigma_2(2i-2,0)$ for $i \in \overline{m+1}$. Let $\sigma\colon \mathcal{L}_m \to X$ be the map which extends σ_1 and σ_2.

From now on, the condition of regularity for X and Y will not be necessary.

1.5 *Proposition*. *Suppose that* $y_0 \in Y$ *and* $f \in S^{X\times Y}$ *are such that* $f|X \times \{y_0\} \sim 1$ *and* $f|\{x\} \times Y \sim 1$ *for each* $x \in X$. *Then* $f \sim 1$.

Proof. Let $h_0\colon X \times \{y_0\} \to \mathbb{R}$ be a map such that $e \circ h_0 = f|X \times \{y_0\}$. For $x \in X$, we take a map $h_x\colon \{x\} \times Y \to \mathbb{R}$ such that $e \circ h_x = f|\{x\} \times Y$ and $h_x(x,y_0) = h_0(x,y_0)$. We define $h\colon X \times Y \to \mathbb{R}$ by $h(x,y) = h_x(x,y)$. Then $e \circ h = f$. We will prove that h is continuous.

We take $(x,y) \in X \times Y$. For $v \in Y$, we choose regions U_v of X and V_v of Y such that $(x,v) \in W_v = U_v \times V_v$ and the diameter of $f(W_v)$ is smaller than 1/4. Then there exists a map $g_v: W_v \to \mathbb{R}$ such that $g_v(x,v) = h(x,v)$ and $e \circ g_v = f|W_v$. Then $g_v|\{x\} \times V_v = h_x|\{x\} \times V_v$. This implies that if w, $v \in Y$, there exists a common extension of g_v and g_w.

Let n be the minimum positive integer for which there exist $v_1, \cdots, v_n \in Y$ such that $y_o \in V_{v_1}$, $y \in V_{v_n}$ and $V_{v_1} \cap V_{v_2} \neq \phi, \cdots, V_{v_{n-1}} \cap V_{v_n} \neq \phi$. Let U be a region of X such that $x \in U \subset U_{v_1} \cap \cdots \cap U_{v_n}$ and let $V = V_{v_1} \cup \cdots \cup V_{v_n}$. Then $(x,y) \in U \times V$ and there exists a map $g: U \times V \to \mathbb{R}$ such that $e \circ g = f$ and g extends each one of the maps $g_{v_i}|U \times V_{v_i}$. Take $(u,v) \in U \times V$. Since $g|\{x\} \times V = h_x|\{x\} \times V$, we have that $g(x,y_o) = h_o(x,y_o)$. This implies that $g|U \times \{y_o\} = h_o|U \times \{y_o\}$, and so $g(u,y_o) = h_u(u,y_o)$. It follows that $g|\{u\} \times V = h_u|\{u\} \times V$. In particular, $g(u,v) = h(u,v)$. Therefore $h|U \times V = g$. This proves that h is continuous and completes the proof.

As a consequence, we obtain the following particular case of Lemma 5 of [8].

1.6 *Corollary. If* $f \in S^X$, *then* $f \sim 1$ *if and only if* $f \approx 1$ *(the constant map* 1*).*

1.7 *Proposition. Let* $\{X_\alpha: \alpha \in J\}$ *be a family of connected, locally connected spaces. For* $p = (p_\alpha) \in \Pi X_\alpha$ *and* $\beta \in J$, *we define* $Y(p,\beta) = \{(x_\alpha) \in \Pi X_\alpha: x_\alpha = p_\alpha$ *for all* $\alpha \neq \beta\}$. *Suppose that* $f: \Pi X_\alpha \to S$ *is a map such that* $f|Y(p,\beta) \sim 1$ *for each* $p \in \Pi X_\alpha$ *and* $\beta \in J$. *Then* $f \sim 1$.

Proof. Fix a point $x = (x_\alpha) \in X = \Pi X_\alpha$. We choose $t \in \mathbb{R}$ such that $e(t) = f(x)$. If L is a subset of J, define $X_L = \Pi\{X_\alpha : \alpha \in L\}$, $x_L^c = (x_\alpha)_{\alpha \notin L} \in X_{J-L}$ and $Y_L = X_L \times \{x_L^c\} \subset X$. From Proposition 1.5 it follows that, for any finite $F \subset J$, there exists a map $g_F: Y_F \to \mathbb{R}$ such that $e \circ g_F = f|Y_F$ and $g_F(x) = t$.

Let $\mathcal{U}$ be the set of basic open subsets $U = \langle U_{\alpha_1}, \cdots, U_{\alpha_n} \rangle$ of X where $U_{\alpha_1}, \cdots, U_{\alpha_n}$ are proper, nonempty regions of $X_{\alpha_1}, \cdots, X_{\alpha_n}$ respectively and the diameter of $f(U)$ is smaller than 1/4. If $U \in \mathcal{U}$, we define $F = \{\alpha_1, \cdots, \alpha_n\}$, and $U_0 = U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \{x_F^c\} \subset U \cap Y_F$. Since $f|U \sim 1$ and U_0 is connected, there exists a map $g_U: U \to \mathbb{R}$ such that $f|U = e \circ g_U$ and $g_U|U_0 = g_F|U_0$.

Let $U = \langle U_{\alpha_1}, \cdots, U_{\alpha_n} \rangle$, $V = \langle V_{\beta_1}, \cdots, V_{\beta_m} \rangle \in \mathcal{U}$ be such that $V \subset U$. We are going to prove that $g_U|V = g_V$. We put $F = \{\alpha_1, \cdots, \alpha_n\}$ and $G = \{\beta_1, \cdots, \beta_m\}$, then $F \subset G$ and $g_F = g_G|Y_F$. We choose a point $y = (y_\alpha) \in V$; we define $S = U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times X_{G-F} \times \{x_G^c\}$, and we define the points $u = (u_\alpha)$ and $z = (z_\alpha)$ by: $z_\alpha = y_\alpha$ if $\alpha \in F$, $z_\alpha = x_\alpha$ if $\alpha \notin F$ and $u_\alpha = y_\alpha$ if $\alpha \in G$, $u_\alpha = x_\alpha$ if $\alpha \notin G$. Then $z \in U_0$, $u \in S \cap V_0$, $S \subset U$ and $S \subset Y_G$, so that $g_U(z) = g_F(z) = g_G(z)$, therefore $g_U|S = g_G|S$. In particular, $g_U(u) = g_G(u)$. This implies that $g_U|V_0 = g_G|V_0 = g_V|V_0$. Hence $g_U|V = g_V$.

From this it follows that if $U, W \in \mathcal{U}$, then $g_U|U \cap W = g_W|U \cap W$. Hence $f \sim 1$.

1.8 *Corollary. Let* $\{X_\alpha: \alpha \in J\}$ *be a family of connected, locally pathwise connected spaces. Suppose that* $f: \Pi X_\alpha \to S$ *is a map and that there exists a point* $x = (\mathbf{x}_\alpha) \in \Pi X_\alpha$ *such that* $f|\mathbf{Y}(\mathbf{x},\beta) \sim 1$ *for all* $\beta \in J$. *Then* $f \sim 1$.

Proof. Let $p = (p_\alpha) \in \Pi X_\alpha$ be any point and let $\beta \in J$. Suppose that $i: X_\beta \to Y(x,\beta)$ and $j: X_\beta \to Y(p,\beta)$ are the inclusions. Since $\Pi\{X_\alpha: \alpha \neq \beta\}$ is pathwise connected, we have that i and j are homotopic (as maps of X_β in ΠX_α). This implies that $f|Y(p,\beta) \sim 1$. Hence $f \sim 1$.

1.9 *Lemma. Let* x *be any point of* X. *Suppose that* X *is locally pathwise connected and that* $f \in S^X$ *is such that* $f \not\sim 1$. *Then there exists a map* $\sigma: S \to X$ *such that* $\sigma(1) = x$ *and* $f \circ \sigma \not\sim 1$.

Proof. Immediate from Theorem 6.1 of [5].

1.10 *Proposition. Suppose that* X *and* Y *are locally pathwise connected. Let* A, B *be closed subsets of* $X \times Y$ *and let* f, $g: X \times Y \to S$ *be maps such that* $X \times Y = A \cup B$, $f|A \sim 1$, $g|A \sim 1$, $f|B \sim 1$ *and* $g|B \sim 1$. *If there exists* $y \in Y$ *such that* $f|X \times \{y\} \not\sim 1$ *and* $g|X \times \{y\} \sim 1$, *then* $g \sim 1$.

Proof. By Proposition 1.5, it is enough to prove that $g|\{u\} \times Y \sim 1$ for each $u \in X$. Suppose that there exists $x \in X$ such that $g|\{x\} \times Y \not\sim 1$. Let $\delta: S \to X \times \{y\}$ and $\lambda: S \to \{x\} \times Y$ be maps such that $f \circ \delta \not\sim 1$, $g \circ \lambda \not\sim 1$ and $\delta(1) = (x,y) = \lambda(1)$. We define $\psi: S \times S \to X \times Y$ by $\psi(s,t) = (P_X(\delta(s)), P_Y(\lambda(t)))$ where P_X and P_Y are the projections of $X \times Y$ in X and Y respectively. Since

$\mathcal{P}(S \times S) = 1$ (Theorem 3, §4 of [3]), we have that there exist integers a, b with $a \neq 0$ or $b \neq 0$ such that $(f \circ \psi)^a (g \circ \psi)^b \sim 1$. Then $(f^a g^b) \circ \psi \circ i \sim 1$ where $i: S \to S \times \{1\}$ is the inclusion, so that $(f \circ \delta)^a (g \circ \delta)^b \sim 1$. Similarly, $(f \circ \lambda)^a (g \circ \lambda)^b \sim 1$. Then $(f \circ \delta)^a \sim 1$, so that $a = 0$. This implies that $(g \circ \lambda)^b \sim 1$, so $b = 0$. This contradiction ends the proof.

2. Main Theorems

2.1 *Theorem. If* $\{X_\alpha : \alpha \in J\}$ *is a nonempty family of connected, locally pathwise connected nonempty spaces, then* $\mathcal{P}(\Pi X_\alpha) = \sup\{\mathcal{P}(X_\alpha) : \alpha \in J\}$.

Proof. It is easy to prove that $\mathcal{P}(\Pi X_\alpha) \geq \sup\{\mathcal{P}(X_\alpha) : \alpha \in J\}$. Suppose that $\mathcal{P}(\Pi X_\alpha) \geq m > \sup\{\mathcal{P}(X_\alpha) : \alpha \in J\}$. Then there exist closed subsets A, B of $X_o = \Pi X_\alpha$ and there exist $f_1, \cdots, f_m \in S^{X_o}$ such that $X_o = A \cup B$, $f_i|A \sim 1$, $f_i|B \sim 1$ for all $i \in \bar{m}$ and $f_1, \cdots, f_m$ are linearly independent. We choose a point $x = (x_\alpha) \in X_o$. By Corollary 1.7, there exists $\beta \in J$ such that $f_1|Y(x,\beta) \not\sim 1$ where $Y(x,\beta) = \{(w_\alpha) \in X_o : w_\alpha = x_\alpha \text{ for all } \alpha \neq \beta\}$. Since $\mathcal{P}(Y(x,\beta)) < m$, there exist integers $a_1, \cdots, a_m$ not all zero such that $f_1^{a_1} \cdot \cdots \cdot f_m^{a_m} | Y(x,\beta) \sim 1$. Applying Proposition 1.9 to $X = X_\beta$ and $Y = \Pi\{X_\alpha : \alpha \neq \beta\}$, we obtain that $f_1^{a_1} \cdot \cdots \cdot f_m^{a_m} \sim 1$. This contradiction completes the proof.

2.2 *Theorem. If* $\{X_\alpha : \alpha \in J\}$ *is a nonempty family of connected, locally pathwise connected, normal nonempty* T_1*-spaces, then* $\mathcal{R}(\Pi X_\alpha) = \sup\{\mathcal{R}(X_\alpha) : \alpha \in J\}$.

Proof. We put $X = \Pi X_\alpha$. It is easy to prove that $\varkappa(X) \geq \sup\{\varkappa(X_\alpha): \alpha \in J\}$. Suppose that $\varkappa(X) \geq m > \sup\{\varkappa(X_\alpha): \alpha \in J\}$. Let A, B, $C_1, \cdots, C_{m+1}$ and $\sigma: \mathcal{L}_m \to X$ be as in Proposition 1.4. We make $\mathcal{L} = (\mathcal{L}_m)^J$ (the product of J copies of $\mathcal{L}_m$) and we define $\psi: \mathcal{L} \to X$ by $\psi((s_\alpha)) = (P_\alpha(\sigma(s_\alpha)))$, where $P_\alpha: X \to X_\alpha$ is the projection. For $s \in \mathcal{L}_m$, we call $\gamma(s)$ the point of $\mathcal{L}$ which has all its coordinates equal to s. Then $\gamma: \mathcal{L}_m \to \mathcal{L}$ is continuous. We make $(x_\alpha) = x = \psi(\gamma(0,0)) = \sigma(0,0) \in C_1$. We can suppose that x is an interior point of C_1. Let $A_1 = \psi^{-1}(A)$ and $B_1 = \psi^{-1}(B)$.

Given $\beta \in J$, we put $Y_\beta = \{(y_\alpha) \in X: y_\alpha = x_\alpha$ for all $\alpha \neq \beta\}$, $A_\beta = A \cap Y_\beta$, $B_\beta = B \cap Y_\beta$ and $C_1^\beta = C_1 \cap Y_\beta, \cdots, C_{m+1}^\beta = C_{m+1} \cap Y_\beta$. Since Y_β is normal, there exists a map $f^\beta: Y_\beta \to \mathcal{L}_m$ such that $f^\beta(A_\beta) \subset \mathcal{L}_m^+$, $f^\beta(B_\beta) \subset \mathcal{L}_m^-$ and $f^\beta(C_i^\beta) \subset \{(2i-2,0)\}$ for each $i \in \overline{m+1}$. For $i \in \overline{m+1}$, we make $f_i^\beta = \ell_i \circ f^\beta: Y_\beta \to S$. We make $T_\beta = \{(s_\alpha) \in \mathcal{L}: s_\alpha = (0,0)$ for all $\alpha \neq \beta\}$, then $\psi(T_\beta) \subset Y_\beta$. Define $g^\beta = f^\beta \circ \psi|T_\beta: T_\beta \to \mathcal{L}_m$.

Let $T = (\cup\{T_\beta: \beta \in J\}) \cup (\psi^{-1}(C_1) \cup \cdots \cup \psi^{-1}(C_{m+1}))$, then T is closed in $\mathcal{L}$. We define $g_o: T \to \mathcal{L}_m$ by $g_o(w) = g^\beta(w)$ if $w \in T_\beta$ and $g_o(w) = (2i-2,0)$ if $w \in \psi^{-1}(C_i)$. Then g_o is continuous, $g_o(T \cap A_1) \subset \mathcal{L}_m^+$ and $g_o(T \cap B_1) \subset \mathcal{L}_m^-$. So that there exists a map $g: \mathcal{L} \to \mathcal{L}_m$ such that $g|T = g_o$, $g(A_1) \subset \mathcal{L}_m^+$ and $g(B_1) \subset \mathcal{L}_m^-$. For $i \in \overline{m}$, we make $g_i = \ell_i \circ g: \mathcal{L} \to S$. Since $\gamma(\mathcal{L}_m^+) \subset A_1$ and $\gamma(\mathcal{L}_m^-) \subset B_1$, we have that $(g \circ \gamma)(\mathcal{L}_m^+) \subset \mathcal{L}_m^+$ and $(g \circ \gamma)(\mathcal{L}_m^-) \subset \mathcal{L}_m^-$. Moreover $g \circ \gamma(2i-2) = 2i-2$ for $i \in \overline{m+1}$. This is enough to assert that $\ell_1 \circ g \circ \gamma, \cdots, \ell_m \circ g \circ \gamma$ are linearly independent (Lemma 1.2 of [6]). This implies that $g_1, \cdots, g_m$ are linearly independent.

By Corollary 1.8, there exists $\beta \in J$ such that $g_1|T_\beta \not\sim 1$. Since $\mathcal{P}(Y_\beta) = \kappa(Y_\beta) < m$, we have that $f_1^\beta,\cdots,f_m^\beta$ are linearly dependent. So there exist integers $a_1,\cdots,a_m$ not all zero such that $(f_1^\beta)^{a_1}\cdot\ldots\cdot(f_m^\beta)^{a_m} \sim 1$; then $(\ell_1^{a_1}\cdot\ldots\cdot\ell_m^{a_m}) \circ f^\beta \circ (\psi|T_\beta) \sim 1$, so that $(g_1^{a_1}\cdot\ldots\cdot g_m^{a_m})|T_\beta \sim 1$. Applying Proposition 1.10, we obtain that $g_1^{a_1}\cdot\ldots\cdot g_m^{a_m} \sim 1$. This contradiction ends the proof.

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