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Shchepin [4,5] introduced the notions of κ-metrizability and capacity as a generalization of metric spaces and locally compact groups, and proved that the κ-metrizability is productive [6]. Bennett, Lewis and Luksic [1] showed that κ-metrizability is equivalent to faithful capacity and, that κ-metrizability is not closed-hereditary. In §2, we give a characterization of a κ-metric space with a compact κ-metrizable extension and, a characterization of when βX is a κ-metrizable extension of a κ-metric space X. Next we give a characterization for a space Y (especially, υX) containing a κ-metric space X as a dense subset to be κ-metrizable extension of X, and prove that if a subspace X of the product L of realcompact κ-metric spaces is either dense or open, then $\upsilon X \subset L$.

In the following, we mean by a space a Tychonoff space and by a κ-metric space a normed κ-metric space (see 1.1 below) and assume familiarity with [2], whose terminology will be used throughout. We denote by C(X) (B(X)) the set of (bounded) real-valued continuous functions defined on X, by R the set of real numbers, by RC(X) the set of regular closed subsets of X, by $\mathcal{U}$ (or $\mathcal{V}$) a free ultrafilter in RC(X), by $\beta X(\upsilon X)$ the Stone-Čech compactification (the real-compactification) of X, by $\beta f(\upsilon f)$ the Stone-extension (the Hewitt-extension) of $f \in B(X)$ $(f \in C(X))$ and by $f_a \downarrow f$ that

$\{f_a; f_a \in C(X)\}$ is a decreasing sequence converging to f pointwise.

1. Definitions and Preliminaries

Definition 1.1. A *κ-metric* $d(x,C)$ on X is a mapping $d: X \times RC(X) \to R$ satisfying the following (K1) ~ (K4):

(K1) $d(x,C) = 0 \iff x \in C$.

(K2) $C \subset D \Rightarrow d(x,C) \geq d(x,D)$ for every $x \in X$.

(K3) $d(x,C)$ is continuous in x for every C.

(K4) $f_a \downarrow f$ for every increasing transfinite sequence $\{C_a\}$ where $f_a(x) = d(x,C_a)$, $f(x) = d(x,D)$ and $D = cl(\cup C_a)$.

Now we consider the following condition such that in (K4) $\{f_a\}$ converges to f uniformly, briefly

(UK4) $f_a \downarrow f$(unif.).

A κ-metric d is said to be *normed* if $d(x,\emptyset) \leq 1$ for every $x \in X$. If d is a κ-metric, then $d(x,C)/(1 + d(x,C))$ is always normed ([5], p. 179), and hence, in the following, we mean by *a κ-metric space* a normed κ-metric space, and by k(X) the set $\{f_C(x); f_C(x) = d(x,C), C \in RC(X)\}$. C and D are *f-separated* [3] if $cl_R f(C) \cap cl_R f(D) = \emptyset$ where C, $D \in RC(X)$ and $f \in C(X)$. D is said to be f_C-*separated* (f_C-*unseparated*) if $\inf f_C(D) > 0$ $(= 0)$ where $f_C \in k(X)$ and $D \in RC(X)$. The following lemma are well known or easily verified.

Lemma 1.2. (1) *If* L *is a κ-metric space with a κ-metric* d *and* $X \subset L$, *then* X *is κ-metrizable in each of the following cases* [5]*:* (i) X *is dense (put* $d_X(x,C) = d(x,cl_L C)$*).* (ii) X *is regular closed (put* $d_X(x,C) = d(x,C)$*).* (iii) X *is open (by* (i) *and* (ii)*).*

(2) *Let* X *be dense in* L, $f_a = g_a|X$ *and* $f = g|X$ *where* g_a, $g \in C(L)$. *Then we have* (i) $C \in RC(L) \Rightarrow X \cap C \in RC(X)$. (ii) $D \in RC(X) \Rightarrow cl_L D \in RC(L)$. (iii) $f_a{\downarrow}f$ (unif.) iff $g_a{\downarrow}g$ (unif). (iv) *If* L *is compact, then* $g_a{\downarrow}g \Rightarrow g_a{\downarrow}g$ (unif.)).

Definition 1.3. Let X be a κ-metric space. We consider the following (RC)- and (SRC)-*conditions*:

(RC) Let $C \cap D = \emptyset$, C, $D \in RC(X)$. If there exists $\mathcal{U} \ni D$ such that every $E \in \mathcal{U}$ is f_C-unseparated, then there exists $\mathcal{V} \ni C$ such that E and F are f_B-unseparated for $E \in \mathcal{U}$, $F \in \mathcal{V}$ and $f_B \in k(X)$.

(SRC) $C \cap D = \emptyset$, $C, D \in RC(X) \Rightarrow D$ is f_C-separated.

The following is easily verified.

Lemma 1.4. (1) *If* X *is a* κ*-metric space,* (SRC) $\Rightarrow$ (RC).

(2) *If* X *is pseudocompact, then* $Z(f) \cap D = \emptyset$ *implies* $inf|f(D)| > 0$ *where* $Z(f) \neq \emptyset$, $f \in C(X)$ *and* $RC(X) \ni D \neq \emptyset$.

(3) *If* X *is pseudocompact and* κ*-metrizable, then* X *satisfies* (SRC).

(4) *Let* X *be dense in* L, $g \in B(L)$ *and* $f = g|X$. *Then we have*

(i) *If* $inf|f(D)| > 0$ *for every* $D \in RC(X)$ *with* $Z(f) \cap D = \emptyset$, *then* $Z(g) = cl_L Z(f)$.

(ii) *If* X *is a* κ*-metric dense subspace satisfying* (SRC) *of* L *and* $f \in k(X)$, *then* $Z(g) = cl_L Z(f)$.

2. Compact and Realcompact κ-Metrizable Extensions

Defintiion 2.1. A space L is said to be a κ*-metrizable extension* of a κ-metric space X with a κ-metric d if X is dense in L and there exists a κ-metric d_L on L such that $d_L(x,D) = d(x, X \cap D)$ for $x \in X$, $D \in RC(L)$ (briefly $d_L|X = d$), and it is easily verified that $d_L(x,D)|X = d(x,C)$ for every $x \in X$ implies $D = cl_L C$.

Theorem 2.2. *Let* X *be a* κ*-metric space. Then there exists a compact* κ*-metrizable extension* L *of* X *iff* X *satisfies* (UK4) *and* (RC).

Proof. $\Rightarrow$). It suffices to show by 1.2(2) that X satisfies (RC). Let $C \cap D = \emptyset$, $D \in \mathcal{U}$ and every $E \in \mathcal{U}$ is f_C-unseparated. L being compact, there exists a point $p \in L - X$ with $\mathcal{U} \to p$. Now suppose $f_C^*(p) = 2h > 0$ where $f_C^*(z) = d_L(z, cl_L C)$, $z \in L$, $d_L|X = d$. Then there exists $E \in \mathcal{U}$ such that $p \in cl_L E$ and $f_C^*(E) > h$, which is impossible because E is f_C-unseparated, and hence $f_C^*(p) = 0$. Take $\mathcal{V}$ such that $C \in \mathcal{V}$ and $\mathcal{V} \to p$. For $f_B \in k(X)$, $E \in \mathcal{U}$, $F \in \mathcal{V}$, we have $cl_R f_B(E) \cap cl_R f_B(F) \ni f_B^*(p)$, which shows that X satisfies (RC).

$\Leftarrow$). Let $a(C) = \sup\{f_C(x); x \in X\}$ and put $I_C = [0,a(C)]$. Then $M = \Pi_C I_C$ is a compact κ-metric space [6, Th. 2] and $\phi: X \to M$ is a homeomorphism of X to $\phi(X)$ where $\phi(x) = (f_C(x))$, $x \in X$. We shall show that $L = cl_M \phi(X)$ is a κ-metrizable extension of X. For $y = (y_C) \in L$, put $f_C^*(y) = y_C$. Obviously f_C^* is continuous. To show $Z(f_C^*) = cl_L C$, it suffices to prove that $f_C^*(p) = 0$ implies $p \in cl_L C$. Suppose that $p \notin cl_L C$. Then there exists $D \in RC(X)$ such that $p \in int_L cl_L D$ and $cl_L D \cap cl_L C = \emptyset$. Take $\mathcal{U}$ such that $D \in \mathcal{U}$ and $\mathcal{U} \to p$. Since $f_C^*(p) = 0$, it is easy to see that every $E \in \mathcal{U}$ is f_C-unseparated. By (RC), there exists $\mathcal{V} \ni C$ such that

E and F are f_B-unseparated for $E \in \mathcal{U}$, $F \in \mathcal{V}$ and $f_B \in k(X)$. Let $\mathcal{V} \to q$. Let $f_B \in k(X)$ and $F \in \mathcal{V}$. $\mathcal{U} \to p$ implies $\cap\{cl_R f_B(E);\ E \in \mathcal{U}\} = f_B^*(p) = p_B$. On the other hand, $\mathcal{A}(F) = \{cl_R f_B(E) \cap cl_R f_B(F);\ E \in \mathcal{U}\}$ is a collection of non-empty closed sets in $[0,1]$ with the finite intersection property. Thus $\cap\mathcal{A}(F) = f_B^*(p)$. Since F is an arbitrary element of $\mathcal{U}$, we have $\cap\{\mathcal{A}(F);\ F \in \mathcal{V}\} = f_B^*(p)$. Since $\cap\{cl_R f_B(E);\ E \in \mathcal{V}\} = f_B^*(q)$, we have $f_B^*(p) = f_B^*(q)$, i.e., $p_B = q_B$ for every $f_B \in k(X)$, and hence $p = q$, a contradiction. Thus we have $Z(f_C^*) = cl_L C$ for $C \in RC(X)$. From this fact and (UK4), it is easily verified that the function d_L defined by $d_L(z,D) = f_C^*(z)$ with $cl_L C = D$ is κ-metric on L and $d_L|X = d$, and hence L is a κ-metrizable extension of X.

Lemma 2.3. *Let* X *be dense in a* κ*-metric space* L, *and* Y *a space containing* X *as a dense subset such that every* $f_C \in k(X)$ *has the continuous extension* f_C^* *on* Y *with* $cl_Y C = Z(f_C^*)$ *where* $f_C(x) = d_L(x, cl_L C)$ *(see* 1.2(1(i))*). Then a continuous mapping* ϕ *from* Y *onto* L *that leaves* X *pointwise fixed is a homeomorphism, so* Y *is a* κ*-metrizable extension of* X.

Proof. Let $p, q \in Y - X$, $p \neq q$ and $\phi(p) = \phi(q) = r \in L - X$. Then there exist $C, D \in RC(X)$ such that $cl_Y C \cap cl_Y D = \emptyset$, $p \in int_Y cl_Y C$, $q \in int_Y cl_Y D$, $f_C^*(p) = 0$ and $f_C^*(q) = 2h > 0$. Thus we have $r \in \phi(cl_Y C) \subset cl_L \phi(C) = cl_L C$ and $d_L(r, cl_L C) = 0$. Similarly, $r \in cl_L D$ and we may assume $d_L(r, cl_L D) > h$, a contradiction, i.e., ϕ is 1-1. By the same method, we obtain the closedness of ϕ, so ϕ is homeomorphic.

As a space Y in Lemma 2.3, we can take the following: $Y = X \cup (\beta X - \cup\{Z(\beta f_C) - cl_{\beta X}C;\ C \in RC(X)\})$. $Z(\upsilon f) = cl_{\upsilon X}Z(f)$ for $f \in C(X)$ [2] implies $\upsilon X \subset Y$.

Theorem 2.4. *If* X *is a* κ*-metric space, then we have*

(1) βX *is a* κ*-metrizable extension of* X *iff* X *satisfies* (SRC) *and* (UK4).

(2) *If* X *is pseudocompact, then* βX *is a* κ*-metrizable extension of* X *iff* X *satisfies* (UK4).

Proof. (1) $\Rightarrow$) If $Z(f) \cap Z(g) = \emptyset$, $f,g \in B(X)$, then $cl_{\beta X}Z(f) \cap cl_{\beta X}Z(g) = \emptyset$[2], and hence X satisfies (SRC) because βX is a κ-metrizable extension of X. On the other hand, X satisfies (UK4) by 2.2.

$\Leftarrow$) Since there exists a compact κ-metrizable extension L of X by 2.2, and X satisfies (SRC), it is easily verified that we can take βX as a space Y in 2.3. On the other hand, there exists a continuous mapping from βX onto L that leaves X pointwise fixed, and hence βX is a κ-metrizable extension of X by 2.3.

(2) From (1) and 1.4(3).

Theorem 2.5. *If a subspace* X *of the product* L *of realcompact* κ*-metric spaces* $\{X_a;\ a \in A\}$ *is either dense or open, then we have*

(1) υX *is a* κ*-metrizable extension of* X *and* $\upsilon X \subset L$.

(2) *If* X *is pseudocompact, then* X *is* C*-*embedded in* L *and* $\beta X \subset L$, *especially, if* X *is dense,* $\beta X = L$.

Proof. (1) (i) X *is dense.* Since L is κ-metrizable [6] and realcompact [2], X is a κ-metric space by 1.2(1), and hence there is a mapping ϕ of υX to L that leaves X pointwise fixed [2]. $X \subset \phi(\upsilon X) \subset L$, so $\phi(\upsilon X)$ is κ-metrizable by 1.2(1), and hence υX is homeomorphic to $\phi(\upsilon X)$ by 2.3. (ii) X *is open.* Apply a similar argument above. (2) follows from the fact that the pseudocompactness of X implies $\beta X = \upsilon X$.

Lemma 2.6. *Let* X *be dense in* L, g_a, $g \in C(L)$ *and* $f_a | f$ *where* $f_a = g_a | X$ *and* $f = g|X$. *Then* $g_a \downarrow g$ *iff* X *satisfies the following:*

(L-K4) *for each* $p \in L - X$ *and each* $f_a \in C(X)$ *with* $f_a \downarrow f \in C(X)$, *there exists* $U \in RC(X)$ *such that* $p \in int_L cl_L U$ *and* $f_a \downarrow f$ *(unif. on* U*).*

Proof. Since the implication $\Leftarrow$) follows from 1.2, we shall show $\Rightarrow$). Let $p \in L - X$ and take $V \in RC(X)$ with $p \in int_L cl_L V$. Let us put $F_a = \{x \in V;\ f_a(x) - f(x) \geq \varepsilon\}$ for every a. Obviously $a > b \Rightarrow F_b \supset F_a$.

Case 1). $F_a = \emptyset$ *for some* a. Put U = V. Thus we have $F_a \neq \emptyset$ for every a.

Case 2). $p \notin cl_L F_a$ *for some* a. Then there exists $W \in RC(X)$ such that $p \in int_L cl_L W$ and $cl_L W \cap cl_L F_a = \emptyset$. Put $U = V \cap W$.

Case 3). $p \in cl_L F_a$ *for every* a. Since $g_a(x) - g(x) = f_a(x) - f(x) \geq \varepsilon$ for every $x \in F_a$, and $p \in cl_L F_a$, we have $g_a(p) - g(p) \geq \varepsilon$, a contradiction.

Using 2.6, the following two theorems are easily verified.

Theorem 2.7. *A space* L *is a* κ*-metrizable extension of a* κ*-metric space* X *iff* X *satisfies* (L-K4) *and every* $f_C \in k(X)$ *has the continuous extension* g *over* L *with* $Z(g) = cl_L C$.

Theorem 2.8. *Let* X *be a* κ*-metrizable space. Then the following are equivalent:*

(1) X *satisfies* (υX-K4).

(2) υX *is a* κ*-metrizable extension of* X.

(3) *There exists a realcompact* κ*-metrizable extension of* X.

Quite recently, we proved the following: (1) If βX is κ-metrizable, then X is pseudocompact. (2) If X is locally compact and $\beta X - X$ is κ-metrizable, then X is pseudocompact. (3) If X is a pseudocompact κ-metric space, and Y is a compact κ-metrizable extension of X, then $\beta X = Y$.

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