ON POINT-PICKING GAMES

by

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1. Introduction

Let X be a topological space, P a property of subsets
of X and α an ordinal then the point-picking game $G_\alpha^P(X)$
is defined as follows:

Two players, I and II, take turns playing. A round
consists of Player I choosing a (non-empty) open set $U \subset X$
and then Player II choosing a point $x \in U$. A round is
played for each ordinal less than α. Player I wins the
game if the set of points picked by Player II has property
P, otherwise Player II wins.

This game was introduced and investigated for the cases
P = D (= dense) and P = SD (= somewhere dense) in [BJ],
where, among other things, reasonable criteria were formu-
lated and proven for Player I to have a winning strategy.
The aim here is to try to do the same for Player II. As
we shall see, the situation is more complex in this case.

Let us put for any space X

$$\delta(X) = \sup\{d(Y): Y \subset X \text{ dense in } X\}.$$

It is straightforward to show that if α is any ordinal
and $\delta(X) > \alpha$ then II $\uparrow$ $G_\alpha^D(X)$. In fact, as A. Berner has
shown in [B], this is exactly when Player II has a so-called
stationary winning strategy for $G_\alpha^D(X)$. Our aim is to
investigate under what circumstances is this obvious and
natural sufficient condition for II to win also necessary.
In fact, we shall only do this for the case in which the
length of the game is ω.

Before we formulate our main result let us recall that, for a T_3 space X, I $\dashv G_\omega^D(X)$ if and only if $\pi(X) = \omega$, see [BJ]. It will also be helpful to keep in mind that, again for a T_3 space X, one always has $\delta(X) \leq \pi(X) \leq 2^{\delta(X)}$. Our main result reads as follows:

Theorem 1. The following two statements are equivalent for any given cardinal $\kappa \leq 2^\omega$:

(1) There exists a (T_3) space X with $\delta(X) = \omega$ and $\pi(X) = \kappa$ such that II $\dashv G_\omega^D(X)$.

(2) The real line R can be written as the union of $\leq \kappa$ nowhere dense subsets.

Before we give the proof of this result let us add a few comments. If $\kappa = \omega$ then this is trivial since both (1) and (2) are false. If $\kappa = 2^\omega$ (the maximum possible value of κ) then (2) of course is valid, hence so is (1), hence we get a ZFC example of a T_3 space X not satisfying $\delta(X) > \omega$ and such that II $\dashv G_\omega^D(X)$. Finally, if $\omega < \kappa < 2^\omega$ then theorem 1 shows that, at least for (T_3) spaces of π-weight κ, it is independent of ZFC whether the condition $\delta(X) > \omega$ is also necessary for Player II to win the game $G_\omega^D(X)$.

2. Proof of Theorem 1

As is well known, see e.g. [K], the negation of statement (2) of theorem 1, i.e. that R is not the union of $\leq \kappa$ nowhere dense sets, is equivalent to MAC_κ, which denotes Martin's axiom for countable posets and $\leq \kappa$ dense sets. Hence the following result yields "one half" of theorem 1.

Theorem 2. Assume MAC_κ and let X be any space with $\delta(X) = \omega$ and $\pi(X) \leq \kappa$. Then Player II does not have a winning strategy for $G_\omega^D(X)$.

Proof. Let $s\colon \tau(X)^{<\omega} \times \tau(X) \to X$ be a strategy for Player II, we shall show that s is not winning. In order to do this we first define, by induction on $n \in \omega$, for every sequence $\sigma \in \omega^n$ a countable dense subset S_σ of X as follows.

To define $S_\emptyset$ we first consider the set $X_\emptyset = \{s(U)\colon U \in \tau(X)\}$ which is dense in X since s is a strategy for Player II, hence we may, by $\delta(X) = \omega$, take $S_\emptyset$ to be a countable dense subset of $X_\emptyset$. Now, let $n \in \omega$ and assume that S_σ has been defined for all $\sigma \in \cup\{\omega^k\colon k \leq n\}$, moreover, for every such σ, $S_\sigma = \{x_m^\sigma\colon m \in \omega\}$ and we have some open sets U_m^σ such that
$$x_m^\sigma = s(U_{\sigma(0)}^{\sigma\upharpoonright 0}, U_{\sigma(1)}^{\sigma\upharpoonright 1}, \cdots, U_{\sigma(\ell-1)}^{\sigma\upharpoonright(\ell-1)}, U_m^\sigma),$$
where $\ell = \ell(\sigma)$ is the length (i.e. domain) of σ. (In what follows, we shall write $U(\sigma,i)$ instead of $U_{\sigma(i)}^{\sigma\upharpoonright i}$.) Now, if $\sigma \in \omega^{n+1}$ then we consider the set
$$X_\sigma = \{s(U(\sigma,0), U(\sigma,1), \cdots, U(\sigma,n), U)\colon U \in \tau(X)\}$$
which is again dense in X hence has a countable dense subset $S_\sigma = \{x_m^\sigma\colon m \in \omega\}$, and, of course, we may choose for each $m \in \omega$ an open set U_m^σ such that $x_m^\sigma = s(U(\sigma,0), U(\sigma,1), \cdots, U(\sigma,n), U_m^\sigma)$. This completes the induction.

The (obviously countable) partial order that we want to apply MAC_κ to is $\omega^{<\omega}$ with the extension of sequences as the partial order. To get the dense sets we first recall that X has a π-base β with $|\beta| \leq \kappa$ and for every $B \in \beta$ we put

$$D_B = \{\sigma \in \omega^{<\omega}: (\exists i < \ell(\sigma))(x_{\sigma\upharpoonright i}^{\sigma(i)} = x(\sigma,i) \in B)\}.$$

Since every S_σ is dense in X it is obvious that D_B is

dense in $\omega^{<\omega}$ for each $B \in \beta$, consequently by MAC_κ there is

a generic branch $\tau \in \omega^\omega$ over the family $\{D_B: B \in \beta\}$.

However, it is immediate from our inductive construc-

tion that $\langle \langle U(\tau,i),x(\tau,i)\rangle: i \in \omega\rangle$ is a play of the game

$G_\omega^D(X)$ in which Player II has followed the strategy s and

still the set $\{x(\tau,i): i \in \omega\}$ is dense in X (because it

intersects every $B \in \beta$), hence s is not winning.

In view of our above remark, it immediately follows

that (1) implies (2) in theorem 1.

To see the converse, we first prove the following

result.

   Theorem 3. Let S with $|S| = \kappa$ be a T_2 space with a

countable base β which admits a function $\phi: \beta \to \mathbf{R}^+ = (0,\infty)$

satisfying the following two properties:

   (i) for every $p \in S$ and every $\epsilon > 0$ there is a $B \in \beta$

such that $p \in B$ and $\phi(B) < \epsilon$;

   (ii) there is a sequence $\{\epsilon_n: n \in \omega\} \subset \mathbf{R}^+$ such that

for every $\{B_n: n \in \omega\} \subset \beta$ if $\phi(B_n) < \epsilon_n$ holds for each

$n \in \omega$ then $S \neq \cup\{B_n: n \in \omega\}$.

   Then 2^κ has a countable dense subspace X for which

II $\uparrow G_\omega^{SD}(X)$ (hence II $\uparrow G_\omega^D(X)$ as well). Note that

$\pi(X) = \kappa$ and $\delta(X) = |X| = \omega$.

   Proof. Let us start by noting that, possibly by

passing to an appropriate subsequence of $\{\epsilon_n: n \in \omega\}$, we

may assume that in (ii) for every $\{B_n: n \in \omega\} \subset \beta$ with

$\phi(B_n) < \varepsilon_n$ we actually have that $S\setminus\cup\{B_n: n \in \omega\}$ is infinite
(or, in fact, uncountable).

The construction of the space X, for convenience we
shall define it as a subspace of 2^S rather than 2^κ, is
standard, see e.g. [H] or [J]. Let $\mathcal{D}$ denote the set of
all functions d such that dom(d) is a finite and disjoint
subset of β and range(d) $\subset$ 2, and for every d $\in \mathcal{D}$ we define
$f_d \in 2^S$ by putting:

$$f_d(p) = \begin{cases} d(B), & \text{if } p \in B \in \text{dom}(d); \\ 0, & \text{if } p \in S\setminus\cup \text{ dom}(d). \end{cases}$$

We then put X = $\{f_d: d \in \mathcal{D}\}$, it is well-known that X is
a (countable) dense subset of 2^S.

We now describe, informally, a winning strategy for
Player II in the game $G_\omega^{SD}(X)$. First, it clearly suffices
to do this only for moves of Player I which are traces of
elementary open sets in 2^S, i.e. have the form [s] = $\{f \in X:$
$s \subset f\}$, where s is any 0-1 function defined on some finite
subset of S.

Suppose that Player I's first move is $[s^0]$ where
dom(s^0) = a^0 = $\{p_i^0: i < n^0\}$. In response to this Player II
first picks a disjoint collection β^0 = $\{B_i: i < n^0\} \subset \beta$
such that $p_i \in B_i$ and $\phi(B_i) < \varepsilon_i$ hold for all $i < n^0$.
This is possible by (i). Then II may pick the element
f_{d^0} of $[s^0]$, where $d^0 \in D$ is given by dom(d^0) = β^0 and
$d^0(B_i)$ = $s^0(p_i)$ for each $i < n^0$. If in the next round the
open set $[s^1]$ is played by I, where dom(s^1) = a^1 =
$\{p_i^1: i < n^1\}$, then II first chooses a disjoint collection
β^1 = $\{B_j: n^0 \leq j < n^0 + n^1\}$ such that $p_i^1 \in B_{n^0+i}$ and

$\phi(B_{n^0+i}) < \varepsilon_{n^0+i}$ for $i < n^1$ and then picks the point

$f_d 1 \in [s^1]$, where $\text{dom}(d^1) = \beta^1$ and $d^1(B_{n^0+i}) = s^1(p_i^1)$ for

$i < n^1$. Continuing in this way, when the play is finished

a sequence $\{B_i : i \in \omega\}$ is generated with the property that

$\phi(B_i) < \varepsilon_i$ for all $i \in \omega$, hence $S \setminus \cup \{B_i : i \in \omega\}$ is infinite.

But clearly every choice $f_d i$ of Player II in this play is

such that $f_d i(p) = 0$ for every $p \in S \setminus \{B_i : i \in \omega\}$, hence

$\{f_d i : i \in \omega\}$ is nowhere dense in 2^S and consequently in X

as well. This shows that the strategy we described is

indeed winning.

An immediate corollary of theorem 3 is that if there

is a set $S \in [\mathbf{R}]^\kappa$ which does not have strong measure 0

then there is a countable dense $X \subset 2^\kappa$ such that II $\uparrow G_\omega^{SD}(X)$.

However, in order to finish the proof of theorem 1 we

shall have to look at some other examples.

Let us consider the Baire space ω^ω with its standard

countable base

$$\beta = \{[s] : s \in \omega^{<\omega}\},$$

the "Baire intervals." For any subset $S \subset \omega^\omega$ we let

$\beta_S = \{S \cap [s] : s \in \omega^{<\omega}\}$ and define $\phi_S : \beta_S \to \mathbf{R}^+$ by

$$\phi_S(S \cap [s]) = \frac{1}{2^{|s|}}.$$

It is obvious that $\langle S, \beta_S, \phi_S \rangle$ always satisfies (i) of

theorem 3. Now, the final link in the proof of Theorem 1

is given by the following (so far unpublished) result of

A. Miller and D. Fremlin [MF]:

Proposition (Miller-Fremlin). *The following two state-*

ments are equivalent for any given cardinal κ:

(A) MAC_κ;

(B) *For every* $S \in [\omega^\omega]^\kappa$ *and every* $\{\varepsilon_n: n \in \omega\} \subset \mathbf{R}^+$
there are $\{B_n: n \in \omega\} \subset \beta_S$ *such that* $\phi(B_n) < \varepsilon_n$ *for all* n
and $S = \cup\{B_n: n \in \omega\}$.

Hence if (2) of theorem 1 holds, i.e. MAC_κ fails, then
by the above proposition there is some $S \in [\omega^\omega]^\kappa$ for which
β_S and ϕ_S satisfy conditions (i) and (ii) of theorem 3,
and consequently there is a countable dense $X \subset 2^\kappa$ for
which II $\uparrow G_\omega^{SD}(X)$. This completes the proof of theorem 1.

Before we conclude, let us mention the following easy
consequence of theorem 1: If $\kappa > \omega$ and MAC_κ holds then the games
$G_\omega^D(X)$ and $G_\omega^{SD}(X)$ are undecided for every T_3 space X satis-
fying $\delta(X) = \omega$ and $\pi(X) = \kappa$ (in particular, for every
countable dense subspace of 2^κ). Moreover, it was shown
in [BJ] that similar "undecided" spaces also exist if one
assumes $\Diamond$. This leads to the following natural question.

Problem. Does there exist, in ZFC, a T_3 space X for
which the game $G_\omega^D(X)$ is undecided?

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