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This paper is largely an outgrowth of work done in [6], and the main theorem in this paper depends heavily on a result in that paper. As a corollary to this theorem, we have that any continuum having only pseudo-arcs as proper subcontinua is locally homeomorphic to the pseudo-arc. Another corollary to this theorem will give us that any such continuum has condition *, i.e., if H is a proper subcontinuum of X, where X is such a continuum, and x and y are points of H, and u is open in X such that $H \subseteq u$, then there is a homeomorphism h such that $h(x) = y$ and $h(z) = z$ for $z \notin u$. Examples of continua with the property that every proper subcontinuum is a pseudo-arc include the pseudo-circle [1], the pseudo-solenoids [4], and some examples due to Ingram [5], and Cook [3], as well as the pseudo-arc itself. The pseudo-arc is the only such continuum that is homogeneous [7].

Problem 14, attributed to Duda, in the list of Continuum Theory Problems [2], is the following: "What additional condition(s) make the following statement true? If X has only chainable proper subcontinua and ( ), then X is either chainable or circularly chainable." With a little extra pushing, the techniques used to prove the main theorem will give an answer (Corollary 12) to this question. For additional background, definitions, terminology, we refer the reader to [6]. We need some lemmas before we get to our main results, but first we would like to thank Jim Rogers for having taken the time and for having the patience to listen to these results and proofs.

Lemma 1. *Suppose* X *is a continuum such that each proper nondegenerate subcontinuum of* X *is chainable. Then if* $x \in X$*, and* u *is an open set in* X *such that* $x \in u \subseteq \bar{u} \neq X$*, and* $\varepsilon > 0$*, there is a finite collection* $\mathcal{C}$ *of mutually exclusive taut open (in* $\bar{u}$*) chains which covers* $\bar{u}$ *such that mesh* $\mathcal{C}^* < \varepsilon$*,* ∂c *is totally disconnected for each* $c \in \mathcal{C}^{**}$*, and each chain essentially covers a nondegenerate continuum.*

Proof. We omit this proof, since it is very straightforward.

Lemma 2. *Suppose that* X *is a continuum and* o *is open in* X *such that* $\bar{o} \neq X$*. If* $U = \{u_1, u_2, \cdots u_m\}$ *is an open taut chain cover of* $\bar{o}$*, considered as space, and* o' *is open in* X *such that* $\bar{o}' \subseteq o$*, then* $U' = \{u_i \in U | \bar{u}_i \subseteq o'\}$ *is a collection of open sets such that* $V = \{C | C$ *is a maximal chain of elements of* $U'\}$ *is a finite collection of taut open chains such that if* $x \in \partial U'^*$*, then* x *is in* ∂u *where* u *is an end link of a chain in* V*, and* $x \notin \overline{\{u_i \in U' | u_i \neq u\}}^*$*. (Possibly* $U' = \phi$*.)*

Proof. This proof is also easy, and is therefore omitted.

Lemma 3. *Suppose* X *is a continuum with the property that every nondegenerate proper subcontinuum is chainable and* $\varepsilon > 0$*. If* u *and* u' *are open such that* $\phi \neq u \subseteq \bar{u} \subseteq u'$*,*

$\bar{u} \neq X$, *there is an open set* o *such that* $\bar{u} \subseteq o \subseteq \bar{o} \subseteq u'$, $\bar{o} \neq X$, *and there is a finite collection* $\mathcal{C}$ *of mutually exclusive taut open (in* $\bar{o}$*) chains such that*

(1) *mesh* $\mathcal{C}^* < \varepsilon$ *and* $\mathcal{C}^{**} = \bar{o}$,

(2) *each chain in* $\mathcal{C}$ *essentially covers a nondegenerate continuum,*

(3) $\partial o \subseteq \{\partial a \mid a$ *is an end link of a chain in* $\mathcal{C}\}^*$, *and each point of* ∂o *is in the closure of exactly one link of* $\mathcal{C}^*$, *and*

(4) ∂o *is totally disconnected.*

Proof. Suppose that o' is an open set in X such that $\bar{u} \subseteq o' \subseteq \bar{o}' \subseteq u'$. Suppose $\varepsilon' > 0$ such that $\varepsilon' < \varepsilon$ and $\varepsilon' < \frac{1}{2} d(\bar{u}, X-o')$. There is a finite collection $\mathcal{C}'$ of mutually exclusive taut open (in $\bar{o}'$) chains which covers $\bar{o}'$, mesh $\mathcal{C}'^* < \varepsilon'$, each chain in $\mathcal{C}'$ essentially covers a continuum, and each link of each chain in $\mathcal{C}'$ has totally disconnected boundary. Let $U = \{c \in \mathcal{C}'^* \mid \bar{c} \subseteq o'\}$. Then U* is open in X, $\bar{u} \subseteq U^* \subseteq \bar{U}^* \subseteq o'$, and there is some $\hat{o}$ open in X such that $\bar{U}^* \subseteq \hat{o} \subseteq \bar{\hat{o}} \subseteq o'$. Let U* = o, and let $\hat{U} = \{C \mid C$ is a maximal chain in U}. Then $\hat{U}$ is a finite collection of taut open (in $\bar{o}'$) chains, mesh $U < \varepsilon'$, and $\partial o \subseteq \{\partial u \mid u$ is an end link of a chain in $\hat{U}\}$, etc.

Let $U' = \{u \cup (\partial o \cap \bar{u}) \mid u \in U\}$, and let $\mathcal{C} = \{C \mid C$ is a maximal chain in U'}. Then $\mathcal{C}$ is a finite collection of taut open (in $\bar{o}$) chains with the desired properties.

Lemma **4.** *Suppose* X *is a continuum and* u *is open in* X *such that* $\phi \neq u \subseteq \bar{u} \neq X$ *and* ∂u *is either degenerate or not connected. If* $0 = \{o_1, \cdots o_m\}$ *is an open (in* $\bar{u}$*) taut* *chain cover of* $\bar{u}$*;* $\partial u \subseteq \partial o_1 \cup \partial o_m - (\cup_{1<i<m} \partial o_i)$*; if* $1 \neq m$*,* $\partial o_1 \cap \partial o_m = \phi$*, and* 0 *essentially covers the nondegenerate continuum* $Q \subseteq u$*, then there is an open (in* $\bar{u}$*) taut chain cover* $0' = \{o_1', \cdots o_{m+1}'\}$ *of* $\bar{u}$ *such that* $\partial u \subseteq \partial o_1' \cup \partial o_{m+1}' - (\cup_{1<i<m} \partial o_i')$*,* $\partial o_1' \cap \partial o_{m+1}' = \phi$*,* 0' *refines* 0*, and* 0' *essentially covers* Q*. (Note: Open means open in* $\bar{u}$*, but boundary (*∂*) means boundary in* X*.)*

Proof. Case 1. Suppose $m = 1$. There is a nondegenerate continuum P in $o_1 = \bar{u}$ such that $P \cap \partial o_1 \neq \emptyset$, but $P \not\subseteq \partial o_1$. There are mutually exclusive closed point sets H and K such that $H \cup K = \partial o_1$, $P \cap H \neq \emptyset$. (If ∂o_1 is degenerate, $K = \emptyset$.) There are points w and y in Q such that $w \neq y$. Suppose $z \in P - \partial o_1 - \{w,y\}$. Then $H \cup \{w\}$, $K \cup \{z,y\}$ are mutually exclusive closed sets, and we can find open sets o_1' and o_2' (in $\bar{u}$) such that $o_1' \cup o_2' = \bar{u}$, $H \cup \{w\} \subseteq o_1' - \bar{o}_2$, $K \cup \{z,y\} \subseteq o_2' - \bar{o}_1'$. Let $0' = \{o_1', o_2'\}$.

Case 2. Suppose $m > 1$. Since $o_1 - o_2$ and $o_2 - o_1$ are mutually exclusive closed (in $o_1 \cup o_2$) sets, there are mutually exclusive open (in $o_1 \cup o_2$) sets A and B such that $o_1 - o_2 \subseteq A$, $o_2 - o_1 \subseteq B$, $\bar{A} \cap \bar{B} = \emptyset$, and there is an open set C in $o_1 \cap o_2$ such that $\bar{C} \cap (o_1 - o_2) = \emptyset$, $\bar{C} \cap (o_2 - o_1) = \emptyset$, and $A \cup B \cup C = o_1 \cup o_2$. Let $o_1' = o_1 \cap A$, $o_2' = C$, $o_3' = B \cap o_2$, and for $n > 3$, $n \leq m+1$, $o_n' = o_{n-1}$. Finally $0' = \{o_1', \cdots o_{m+1}'\}$.

Lemma 5. *Suppose* X *is a continuum with the property that every proper subcontinuum of* X *is chainable and* $\varepsilon > 0$*. If* u *and* u' *are open such that* $\emptyset \neq u \subseteq \bar{u} \subseteq u' \subseteq \bar{u}$ *,* $\bar{u} \neq X$*, there is an open set* o *such that* $\bar{u} \subseteq o \subseteq \bar{o} \subseteq u'$*,* $\bar{o} \neq X$*, and* *there is a finite collection* $\mathcal{C}$ *of mutually exclusive taut open (in* $\bar{o}$*) chains such that*

(1) *mesh* $\mathcal{C}^* < \varepsilon$ *and* $\mathcal{C}^{**} = \bar{o}$*,*

(2) *each chain in* $\mathcal{C}$ *essentially covers a nondegenerate continuum,*

(3) $\partial o \subseteq \{\partial a | a$ *is an end link of a chain in* $\mathcal{C}\}^*$*, and each point of* ∂o *is in the closure of exactly one link of* $\mathcal{C}^*$*,*

(4) ∂o *is totally disconnected,*

(5) *if* C *is a chain in* $\mathcal{C}$ *and* c_o *and* c_n *denote the end links of* C*, either* (a) *there is a continuum* P *in* C* *such that* $P \cap \partial o \cap \partial c_o \neq \emptyset$ *and* $P \cap \partial o \cap \partial c_n \neq \emptyset$*,* (b) *each continuum* P *in* C* *fails to intersect* $\partial o \cap \partial c_o$*, or* (c) *each continuum* P *in* C* *fails to intersect* $\partial o \cap \partial c_n$*, and*

(6) *each chain in* $\mathcal{C}$ *contains at least* 6 *links.*

Proof. We can use Lemma 3 to obtain an open set o such that $\bar{u} \subseteq o \subseteq \bar{o} \subseteq u'$, $\bar{o} \neq X$, and a finite collection $\mathcal{D}$ of mutually exclusive taut open (in $\bar{o}$) chains satisfying properties 1-4. Next we can modify the collection $\mathcal{D}$ to obtain the collection $\mathcal{D}'$ of mutually exclusive taut open (in $\bar{o}$) chains satisfying properties 1-5, as follows: Suppose the chain $D = \{d_0, \cdots, d_m\} \in \mathcal{D}$. If there is some continuum P such that $P \subseteq D^*$ and $P \cap \partial d_o \cap \partial o \neq \emptyset$, $P \cap \partial d_m \cap \partial o \neq \emptyset$, then $D \in \mathcal{D}'$. Otherwise, consider $A = \{Q | Q \text{ is a component of } D^* \text{ and } Q \cap \partial d_o \cap \partial o \neq \emptyset\}^*$ and $B = \{Q | Q \text{ is a component of } D^* \text{ and } Q \cap \partial d_m \cap \partial o \neq \emptyset\}^*$. Then A and B are mutually exclusive closed sets such that $A \cup B = D^*$. (It is possible that $A = \emptyset$ or $B = \emptyset$.) But A and B are also open in $\bar{o}$, so we can split D into $D_A = \{d_o \cap A, d_1 \cap A, \cdots, d_m \cap A\}$ and $D_B = \{d_o \cap B, d_1 \cap B, \cdots, d_m \cap B\}$. Replace D in $\mathcal{D}$ with D_A (provided $A \neq \emptyset$, and after we have eliminated any links in the listing above which are empty or nonessential and "closed up" any link adjacent to a nonessential link), and D_B (provided $B \neq \emptyset$, etc.) in $\mathcal{D}'$.

Finally, it is easy to get $\mathcal{C}$ from $\mathcal{D}'$, with all the properties we wanted, by simply applying Lemma 4 now.

Lemma 6. *Suppose* X *is a continuum with the property that every nondegenerate proper subcontinuum of* X *is chainable. If* u *and* u' *are open sets in* X *such that* $\phi \neq u \subseteq \bar{u} \subseteq u' \subseteq \bar{u}' \neq X$, *there is an open set* o *such that* $\bar{u} \subseteq o \subseteq \bar{o} \subseteq u'$ *and for each* $i \in N$, *there is a finite collection* $\mathcal{C}_i$ *of mutually exclusive taut open (in* $\bar{o}$*) chains such that*

(1) *mesh* $\mathcal{C}_i^* < \frac{1}{i}$ *and* $\mathcal{C}_i^{**} = \bar{o}$,

(2) $\mathcal{C}_i^*$ *is an amalgamation of* $\mathcal{C}_{i+1}^*$,

(3) *each chain in* $\mathcal{C}_i$ *essentially covers a continuum and contains at least* 5 *links,*

(4) $\partial o \subseteq \{\partial A | A$ *is an end link of a chain in* $\mathcal{C}_i^*\}$ *and each point of* ∂o *is in the closure of exactly one link of* $\mathcal{C}_i^*$,

(5) *if* C *is a chain in* $\mathcal{C}_i$ *and* c_o *and* c_n *denote the end links of* C, *either* (a) *there is a continuum* P *in* C* *such that* $P \cap \partial o \cap \partial c_o \neq \emptyset$ *and* $P \cap \partial o \cap \partial c_n \neq \emptyset$, (b) *each continuum* P *in* C* *fails to intersect* $\partial o \cap \partial c_o$, *or* (c) *each continuum* P *in* C* *fails to intersect* $\partial o \cap \partial c_n$; *and*

(6) *each link of each chain in* $\mathcal{C}_i$ *is regular, i.e.,* $\text{Int}_{\bar{o}}\, \bar{c} = c$ *for* $c \in \mathcal{C}_i^*$.

Proof. Suppose $\varepsilon_1 < 1/8$. From Lemma 5, there are an open set o_1, and a finite collection $\mathcal{D}_1$ of chains satisfying the properties guaranteed there (for ε_1, u, u'). Let $a_1 = \min\{d(e,e') | e$ and e' are nonadjacent links of $\mathcal{D}_1^*\}$, $\mathcal{E}_1 = \{d \in \mathcal{D}_1^* | d \cap \partial o_1 = \emptyset\}$, $e_1 = d(\mathcal{E}_1^*, X - o_1)$.

Choose $\varepsilon_2 > 0$ such that (1) $\varepsilon_2 < 1/16$, (2) $\varepsilon_2 < 1/4(\varepsilon_1)$, (3) $\varepsilon_2 < \frac{1}{4} e_1$, (4) $\varepsilon_2 < \frac{1}{4} a_1$, and (5) $\varepsilon_2 < \frac{1}{4} d(\bar{o}_1, X-u')$. Then use Lemma 5 to find an open set o_2 and a finite collection $\mathcal{D}_2$ of chains satisfying the properties guaranteed there for ε_2, o_1, u'. Assume, without loss of generality, that if $d \in \mathcal{D}_2^*$, then $d \cap \bar{o}_1 \neq \emptyset$. Then $\bar{o}_1 \subseteq o_2 \subseteq \bar{o}_2 \subseteq u'$ and $d(o_2, X-u') \geq 3/4\, d(\bar{o}_1, X-u')$. Choose $\varepsilon_3 > 0$ such that (1) $\varepsilon_3 < \frac{1}{32}$, (2) $\varepsilon_3 < \frac{1}{4} \varepsilon_2$, (3) $\varepsilon_3 < \frac{1}{4} e_2$ where $e_2 = d(\mathcal{E}_2^*, X-o_2)$ and $\mathcal{E}_2 = \{d \in \mathcal{D}_2^* | d \cap \partial o_2 = \emptyset\}$, (4) $\varepsilon_3 < \frac{1}{4} a_2$ where $a_2 = \min\{d(e,e') | e$ and e' are nonadjacent links of $\mathcal{D}_2^*\}$, and (5) $\varepsilon_3 < \frac{1}{8} d(o_2, X-u')$. Then use Lemma 5, etc. Continue this process, obtaining for each i a finite collection $\mathcal{D}_i$ of mutually exclusive taut open (in $\bar{o}_i$) chains such that (1) mesh $\mathcal{D}_i^* < \varepsilon_i$ and $\bar{o}_{i-1} \subseteq \mathcal{D}_i^{**} = \bar{o}_i$, (2) each chain in $\mathcal{D}_i$ essentially covers a continuum and contains at least 6 links, and (3) $\partial o_i \subseteq \{\partial a | a$ is an end link of a chain in $\mathcal{D}_i\}$, etc. (Choose ε_i analogously to the way ε_3 was chosen.)

Now for $d \in \mathcal{D}_i^*$, form $d^\#$ as follows: Let $d^1 = \{e \in \mathcal{D}_{i+1}^* | e \cap \bar{d} \neq \phi\}^*$, $d^2 = \{e \in \mathcal{D}_{i+2}^* | e \cap \bar{d}^1 \neq \phi\}^*$, $\cdots$, $d^n = \{e \in \mathcal{D}_{i+n}^* | e \cap \overline{d^{n-1}} \neq \phi\}^*$, and let $d^\# = \cup_{j=1}^\infty d^j$. Let $\mathcal{C}_i^! = \{d^\# | d \in \mathcal{D}_i^*\}$, and let $o = \mathcal{C}_i^{!*}$. (Note that o is <u>not</u> ambiguously defined.) Then $\bar{u} \subseteq o \subseteq \bar{o} \subseteq u'$, and we can let

$\mathcal{C}_i^{\#} = \{d^{\#} \cup (\bar{d}^{\#} \cap \partial o) | d^{\#} \in \mathcal{C}_i^{!}\}$, and $\mathcal{C}_i = \{C | C$ is a maximal chain formed of members of $\mathcal{C}_i^{\#}\}$.

Suppose $x \in \partial o$. Then $x \in c$ for some $c \in \mathcal{C}_i^{\#}$, and $c = d^{\#} \cup (\bar{d}^{\#} \cap \partial o)$ for some $d \in \mathcal{D}_i$. Suppose there is $c' \neq c$ in $\mathcal{C}_i^{\#}$ such that $x \in c'$. Now $c' = d'^{\#} \cup (\overline{d'^{\#}} \cap \partial o)$ for some $d' \in \mathcal{D}_i$. Then $d(x,d) < 2\ \varepsilon_{i+1}$ and $d(x,d') < 2\ \varepsilon_{i+1}$, and d and d' must be adjacent links in $\mathcal{D}_i^*$ because of property (4) in the choice of ε_j's. Because of property (3), both d and d' are end links, and $\{d,d'\}$ is a maximal chain formed of elements of $\mathcal{D}_i$. But each such chain has at least 6 links, so this can't happen. Then $x \in c$ for exactly one link c of $\mathcal{C}_i^*$, and that link (property (3) in the choice of ε_i's) must be an end link.

It is routine to check that for each i, $\mathcal{C}_i$ is a finite collection of mutually esclusive taut open (in $\bar{o}$) chains such that (1) mesh $\mathcal{C}_i^* < 3\ \varepsilon_i < \frac{3}{2^{i+2}} < \frac{1}{2^i} < \frac{1}{i}$ and $\mathcal{C}_i^{**} = \bar{o}$, and (2) $\partial o \subseteq \{\partial a | a$ is an end link of a chain in $\mathcal{C}_i\}^*$ and if $x \in \partial o$, then x is in the closure of exactly one link of $\mathcal{C}_i^*$. Also, each chain in $\mathcal{C}_i$ has at least 6 links. What about whether or not each chain essentially covers a continuum? Suppose that D is a chain in $\mathcal{D}_i$ and $D^{\#}$ denotes the corresponding chain in $\mathcal{C}_i$. Denote D by $\{d_o,\cdots d_m\}$, so that $D^{\#} = \{d_o^{\#},\cdots,d_m^{\#}\}$. If there is some continuum P such that $P \subseteq D^*$, $P \cap \partial d_o \cap \partial o_i \neq \phi$ and $P \cap \partial d_m \cap \partial o_i \neq \phi$, then $D^{\#}$ essentially covers P. Otherwise let $A = \{Q | Q$ is a component of D^* and $Q \cap \partial d_o \cap \partial o_i \neq \phi\}^*$, and $B = \{Q | Q$ is a component of D^* and $Q \cap \partial d_m \cap \partial o_i \neq \phi\}^*$. Either $A = \phi$ or $B = \phi$, say $B = \phi$. Then $D^{\#} = \{d_o^{\#},\cdots,d_m^{\#}\}$ could possibly contain a nonessential link, namely $d_m^{\#}$. (Note that $d_m^{\#} \cap \partial o = \phi$.) Modify $\mathcal{C}_i$ by throwing out the link $d_m^{\#}$ in $D^{\#}$, and replacing $d_{m-1}^{\#}$ with $d_{m-1}^{\#} \cup (\overline{d_{m-1}^{\#}} \cap d_m^{\#})$.

At this point, all properties except perhaps numbers 5 and 6 have been satisfied. But, it is an easy matter to satisfy property 6. We can simply replace any link c in $\mathcal{C}_i$ with $\mathrm{Int}_{\overline{o}}(\overline{c})$ if $c \subsetneq \mathrm{Int}_{\overline{o}}(\overline{c})$. To satisfy property 5, we may still have to "split" some chains in $\mathcal{C}_i$, as we did in the last proof. Then the $\mathcal{C}_i$'s satisfy all properties required.

Lemma 7. *Suppose that* X *is a continuum with the property that every nondegenerate proper subcontinuum of* X *is a pseudo-arc. If* u *and* u' *are open sets in* X *such that* $\phi \neq u \subseteq \overline{u} \subseteq u' \subseteq \overline{u'} \neq X$, *there are open sets* o *and* o' *in* X *such that* $\overline{u} \subseteq o \subseteq \overline{o} \subseteq o' \subseteq \overline{o'} \subseteq u'$ *and for each* $i \in N$, *there are collections* $\mathcal{C}_i$ *and* $\mathcal{C}_i'$ *of mutually exclusive taut open (in* $\overline{o}$ *and* $\overline{o'}$, *respectively) chains such that*

(1) *Mesh* $\mathcal{C}_i'^* < \frac{1}{i}$, $\mathcal{C}_i'^{**} = \overline{o'}$, $\mathcal{C}_i^{**} = \overline{o}$, *and* $\{\mathrm{Int}(d) \mid d \in \mathcal{C}_i^*\}$ *refines* $\mathcal{C}_i'^*$;

(2) $\mathcal{C}_i'^*$ *is an amalgamation of* $\mathcal{C}_{i+1}'^*$, $\mathcal{C}_i^*$ *is an amalgamation of* $\mathcal{C}_{i+1}^*$, *and* $\mathcal{C}_i^*$ *is an amalgamation of* $\{c' \cup (\overline{c'} \cap \partial o) \mid c' \in \mathcal{C}_j'^* \text{ and } c' \subseteq o\}$ *for some* j;

(3) *for each chain* $C \in \mathcal{C}_i$, *there is a continuum* P *such that* (A) $P - C^*$ *has a limit point in* c *if and only if* c *is an end link of* C, (B) *if* Q *is a subcontinuum of* P *and* Q *intersects the boundary of* o *in both end links of* C, *then* $Q \not\subseteq X - \mathrm{Int}\, C^*$, *and* (C) $P \cap \overline{o} \subseteq C^*$;

(4) $\partial o' \subseteq \{\partial a | a$ *is an end link of a chain in* $\mathcal{C}_i'\}^*$ *and each point of* $\partial o'$ *is in the closure of exactly one link of* $\mathcal{C}_i'^*$;

(5) $\partial o \subseteq \{\partial a | a$ *is an end link of a chain in* $\mathcal{C}_i\}^*$ *and each point of* ∂o *is in the closure of exactly one link of* $\mathcal{C}_i^*$;

(6) *each chain in* $\mathcal{C}_i'$ *contains at least* 5 *links and each chain in* $\mathcal{C}_i$ *contains at least* 2 *links.*

Proof. From Lemma 6 we know that there is an open set o' such that $\bar{u} \subseteq o' \subseteq \bar{o}' \subseteq u'$ and for each i, there is a finite collection $\mathcal{C}_i'$ of mutually exclusive taut open (in $\bar{o}'$) chains such that (1) mesh $\mathcal{C}_i'^* < \frac{1}{i}$ and $\mathcal{C}_i'^{**} = \bar{o}'$, (2) $\partial o' \subseteq \{\partial a | a$ is an end link of a chain in $\mathcal{C}_i'\}^*$ and each point of $\partial o'$ is in the closure of exactly one link of $\mathcal{C}_i'^*$, (3) each chain in $\mathcal{C}_i'$ essentially covers a continuum and contains at least 5 links, (4) for each i, $\mathcal{C}_i'^*$ is an amalgamation of $\mathcal{C}_{i+1}'^*$, (5) if C is a chain in $\mathcal{C}_i'$, either (a) there is a continuum P in C^* such that $P \cap \partial o' \cap \partial c \neq \emptyset$ iff c is an end link of c, or (b) there is one end link c of C such that $\partial c \cap \partial o' = \emptyset$, and (6) $\mathrm{Int}_{\bar{o}'}\bar{c} = c$ for each c in $\mathcal{C}_i'^*$.

Let $\alpha = d(X-o', \bar{u})$. There is i_o such that $\frac{1}{i_o} < \frac{1}{4}\alpha$. For each i let D_i denote the collection of all maximal subchains formed from $\{c \cup (\partial o \cap \bar{c}) | c \in \mathcal{C}_{i_o+i}'^*$ and $c \subseteq c'$ for some $c' \in \mathcal{C}_{i_o}'^*$ such that $c' \cap \partial o' = \phi\}$, where $o = \{c' \in \mathcal{C}_{i_o}'^* | c' \cap \partial o' = \phi\}^*$.

Then $\bar{u} \subseteq o \subseteq \bar{o} \subseteq o'$, and the sequence $D_1, D_2, \cdots$ has properties 1, 2, 3, and 4 (with respect to o) in the statement of Lemma 6, except that some chains in D_i^* may contain fewer than 5 links. Exactly one of the following is true about $D = \{d_o, \cdots, d_m\} \in D_i$: (a) some component P of D^* intersects both $\partial o \cap d_o$ and $\partial o \cap d_m$, (b) $\partial o \cap d_o = \emptyset$ or (c) $\partial o \cap d_m = \emptyset$.

Form the collection $\mathcal{C}_1$ of mutually exclusive taut open (in $\bar{o}$) chains as follows: For each $D = \{d_o, \cdots d_m\} \in D_i$, there is a nondegenerate continuum $P_D \subset \bar{o}$, which is essentially covered by D, and we can let Q_D denote the component of $\bar{o}'$ containing P_D. (*Case* 1.) If there is a continuum $T \subseteq D^*$ such that for some sets E_1 and E_2 in o', $\bar{E}_1 \cap \bar{E}_2 = \phi$, containing $\partial o \cap d_o$ and $\partial o \cap d_m$, respectively, then T', the component of $E_1 \cup D^* \cup E_2$ containing T, intersects both $E_1 - \bar{D}^*$ and $E_2 - \bar{D}^*$, and $m \neq 0$, leave D as it is, that is, it becomes a chain in $\mathcal{C}_1$. (Note that if there is a continuum $P \subseteq D^*$ such that $P \cap d_o \cap \partial o \neq \phi$ and $P \cap d_m \cap \partial o \neq \emptyset$ and $m \neq 0$, then case 1 occurs.) (*Case* 2). If $m = 0$, choose $z \in P_D - \partial o$. There is some j such that $z \in \bar{d}$ for $d \in D_j^*$ implies $\bar{d} \cap \partial o = \phi$. Let $A = \{\hat{d} \in D_j^* | z \notin \bar{\hat{d}}, \hat{d} \subseteq D^* = d_o\}^*$ and $B = \{d' \in D_j^* | d' \subseteq D^* \text{ and } z \in \bar{d}'\}^*$. Now we can find mutually exclusive open sets U and V such that $A = U \cup V$; there is a continuum $T \subseteq \{U,V,B\}^*$ such that if E_1 and E_2 are open sets in o_1, $\bar{E}_1 \cap \bar{E}_2 = \phi$, and E_1 and E_2 contain $U \cap \partial o$ and $V \cap \partial o$, respectively, then T', the component of $D^* \cup E_1 \cup E_2$ containing T, intersects both $E_1 - \bar{o}$ and $E_2 - \bar{o}$; and U and V are both unions of some links of $D_{j'}^*$ for some $j' > j$. Replace D with the chain $\{U,B,V\}$. (*Case* 3). If $d_o \cap \partial o = \emptyset$, choose $z \in P_D - (D - \{d_o\})^*$. There is some j such that $z \in \bar{d}$ for $d \in D_j^*$

implies $\bar{d} \subseteq d_o$. Let $A = \{d \in D_j^* | z \notin \bar{d}, d \subseteq D^*\}^*$ and $B = \{d' \in D_j^* | z \in \bar{d}', d' \subseteq D^*\}^*$. Now we can find mutually exclusive open sets U and V such that $A = U \cup V$; there is a continuum $T \subseteq \{U,B,V\}^*$ such that if E_1 and E_2 are open sets in o' such that $\bar{E}_1 \cap \bar{E}_2 = \phi$ and E_1 and E_2 contain $U \cap \partial o$ ($\neq \emptyset$), and $V \cap \partial o$ ($\neq \emptyset$), respectively, then T', the component of $D^* \cup E_1 \cup E_2$ containing T, intersects both $E_1 - \bar{o}$ and $E_2 - \bar{o}$; and U and V are both unions of some links of $D_{j'}^*$, for some $j' > j$. Replace D with $\{U \cap d_m, U \cap d_{m-1}, \cdots, U \cap d_1, d_o, V \cap d_1, V \cap d_2, \cdots, V \cap d_m\}$. (*Case* 4.) If every component of D^* fails to intersect $d_m \cap \partial o$, we can follow a procedure analogous to the one in Case 3, so we leave it out. We will let $\mathcal{C}_1$ denote the resulting collection of chains.

In order to construct $\mathcal{C}_2$, we must first modify D_2 to $\hat{D}_2$. If $D \in D_2$, $D^* \subseteq D'^*$ for some $D' \in D_1$. But D' may have been changed. If $D' \in \mathcal{C}_1$, then $D \in \hat{D}_2$. Otherwise D' has become some $\hat{D} \in \mathcal{C}_1$. Recall however that each link of $\hat{D}$ is a union of some links of $D_{j'}^*$, for some j'. Replace D with $\{E \in D_{j'} | E^* \subseteq D^*\}$ in $\hat{D}_2$. Now construct $\mathcal{C}_2$ from $\hat{D}_2$ as we constructed $\mathcal{C}_1$ from D_1. Next modify D_3 to $\hat{D}_3$, analogously to the way we modified D_2 to $\hat{D}_2$, and construct $\mathcal{C}_3$ from $\hat{D}_3$. Continue this process, constructing the desired sequence $\mathcal{C}_1, \mathcal{C}_2, \cdots$.

Finally we are ready for the theorems. The theorem that follows is a generalization of Theorem 7 of [6], the statement of which follows:

Theorem 7 [6]. *Suppose* X *is the pseudo-arc and* Y *is the pseudo-circle. Then there are open sets* o *of* X *and* u *of* Y *and a homeomorphism* $h: \bar{o} \to \bar{u}$ *such that* $h(\partial o) = \partial u$.

The idea of the proof of Theorem 7 of [6] is the following: A sequence $\hat{C}(1), \hat{C}(2), \cdots$ of open covers of $\bar{o}$ such that each $\hat{C}(i)$ consists of a finite collection of taut maximal chains is constructed; and a sequence $\hat{D}(1), \hat{D}(2), \cdots$ of open covers of $\bar{u}$ such that each $\hat{D}(i)$ consists of a finite collection of taut maximal chains is constructed. Further, for each i, there is a pattern f_i such that $\hat{C}(i+1)$ follows f_i in $\hat{C}(i)$ and $\hat{D}(i+1)$ follows f_i in $\hat{D}(i)$. Also, the sequences $\hat{C}(1), \hat{C}(2), \cdots$ and $\hat{D}(1), \hat{D}(2), \cdots$ are carefully constructed so that Theorem 6 of [6] can be applied to conclude that $\bar{o}$ is homeomorphic to $\bar{u}$.

Each chain C in C(i) (Theorem 7 of [6]) has the following needed properties:

(A) Both end links of C intersect ∂o.

(B) If a link of C is not an end link, then its closure does not intersect ∂o.

(C) There is a pseudo-arc that "runs all the way through C, and hangs out both of the ends."

Likewise, each chain D in D(i) has the same properties in $\bar{u}$. These features of these chains, along with the fact that for both continua, all proper subcontinua are pseudo-arcs, are the ones that allow us to conclude finally that $\bar{o}$ and $\bar{u}$ are homeomorphic in a nice way. Of course, a lot more details and proof are necessary.

What we have done with Lemma 7 of this paper is to set the same stage. In that lemma, $\overline{o}$ and the sequence $\mathcal{C}(1), \mathcal{C}(2), \cdots$ play the same roles as do $\overline{o}$ and the sequence $\hat{C}(1), \hat{C}(2), \cdots$, or $\overline{u}$ and the sequence $\hat{D}(1), \hat{D}(2), \cdots$. The set $\overline{o}'$ and the sequence $\mathcal{C}'(1), \mathcal{C}'(2), \cdots$ are present only to form the proper background for the construction of $\overline{o}$ and the sequence $\mathcal{C}(1), \mathcal{C}(2), \cdots$. So, finally, Lemma 7 tells us that the proof given for Theorem 7 in [6] also proves the following theorem, if we make some minor changes in that proof at the appropriate points. (We will be able to carve $\overline{o}$ and $\overline{v}$ up into the desired pieces (the G_i's of Theorem 7 of [6]), set up the patterns we need, etc.). Since that proof is rather long and technical, we do not include it here.

Theorem 8. *Suppose* X *is a nondegenerate continuum with the property that every nondegenerate proper subcontinuum of* X *is a pseudo-arc. If* $\phi \neq u \subseteq \overline{u} \subseteq u' \subseteq \overline{u}' \neq X$ *and* u *and* u' *are open sets in* X, *there are an open set* v *in* X *and an open set* o *in* P, *the pseudo-arc, such that* (1) $\overline{u} \subseteq v \subseteq \overline{v} \subseteq u'$, *and* (2) *there is a homeomorphism* $h: \overline{o} \to \overline{v}$ *such that* $h(\partial o) = \partial v$.

Corollary 9. *If* X *is a nondegenerate continuum with the property that all proper, nondegenerate subcontinua of* X *are pseudo-arcs, then* X *has condition* *.

Proof. The pseudo-arc has condition * [8]. We can apply the previous theorem then to get this result.

Theorem 10. *Suppose that* X *is a continuum, and every proper subcontinuum of* X *is chainable. Suppose also that* X *has the property that whenever* $0 = \{0_1, \cdots 0_n\}$ *is a finite collection of open sets in* X *such that* (1) $\overline{0}_i \cap \overline{0}_j = \phi$ *for* $i \neq j$, *and* (2) *each component of* $X - 0^*$ *intersects boundaries of no more than two elements of* 0, *then there is a taut (circular) chain cover* $\mathcal{C}$ *of* X *such that each* 0_i *is a union of some links of* $\mathcal{C}$. *Then* X *is (circularly) chainable.*

Proof. We prove the theorem for the chainable case, and leave the other case to the reader. In the following, if $\varepsilon > 0$ and $x \in X$, $D_\varepsilon(x)$ will denote the ε-neighborhood of x in X. Likewise, if $A \subseteq X$, $D_\varepsilon(A)$ will denote the ε-neighborhood of A.

Suppose that $\varepsilon > 0$ and $x \in X$ such that $\overline{D_\varepsilon(x)} \neq X$. Let $v' = X - \overline{D_{\varepsilon/8}(x)}$. There is an open set u in X such that $\overline{v}' \subseteq u \subseteq \overline{u} \neq X$ and such that there is a finite collection $\mathcal{E}$ of mutually exclusive taut open (in $\overline{u}$) chains such that (1) mesh $\mathcal{E}^* < \varepsilon/8$ and $\mathcal{E}^{**} = \overline{u}$; (2) each chain in $\mathcal{E}$ essentially covers a continuum; (3) $\partial u \subseteq \{\partial c \mid c$ is an end link of a chain in $\mathcal{E}\}^*$, and each point of ∂u is in the closure of exactly one link of the chain that contains it; and (4) each chain in $\mathcal{E}$ has an odd number of links and has at least 3 links.

Let $\hat{\mathcal{D}} = \{c \cap u \mid c \in \mathcal{E}^*\}$ and let $\mathcal{D} = \{D \mid D$ is a maximal chain composed of elements of $\hat{\mathcal{D}}\}$. There is an open set v in X such that $\overline{v} \subseteq X - \overline{u}$ and if P is a component of $X - (v \cup \mathcal{D}^{**})$, then P does not intersect the closure of more than one member of $\mathcal{D}^*$, and that member must be an end link. For each $D \in \mathcal{D}$, let $\{D_o, D_1, \cdots D_{m_D}\}$ denote a listing of the links of D. Let $0 = \{0_1, \cdots 0_n\}$ denote a listing of the elements of $\{v\} \cup \{D_i | i \text{ is even}, D \in \mathcal{D}\}$. Then (1) $\overline{0}_i \cap \overline{0}_j = \phi$ for $i \neq j$, and (2) each component of $X - 0^*$ intersects the boundaries of no more than 2 elements of 0, and so there is a taut open chain cover $\mathcal{C}$ of X such that each 0_i is a union of some links of $\mathcal{C}$.

We will use this chain to construct an open taut chain cover of mesh less than ε: Let $F = \{E^* | E \text{ is a maximal subchain of } \mathcal{C} \text{ such that } E^* \subseteq 0^*\}$. Replace $\mathcal{C}$ with $\hat{\mathcal{C}} = F \cup \{d \in \mathcal{C} | d \not\subset 0^*\}$. Note that mesh $F < \varepsilon/8$ and $\hat{\mathcal{C}}$ is an open taut chain cover of X. If $d \in \hat{\mathcal{C}}$ and $d \not\subset 0^*$, then d can intersect only 2 members of 0, for otherwise $d \cap 0_i \neq \phi$, $d \cap 0_j \neq \phi$, $d \cap 0_k \neq \phi$, $\{i,j,k\}$ a 3-member set, ıeans there are $d_1 \subseteq 0_i$, $d_2 \subseteq 0_j$, and $d_3 \subseteq 0_k$ such that d_1, d_2, d_3 are links of $\hat{\mathcal{C}}$ and $d \cap d_i \neq \phi$ for $i \in \{1,2,3,\}$. But then d intersects 3 links of $\hat{\mathcal{C}}$, which can't be.

Let $E_1 = \{d \in \mathcal{D}^* | \overline{d} \cap \partial u = \phi\}$ and let $0' = \{D_i | i \text{ is odd}, D_i \in \mathcal{D}^*\}$. Choose $\alpha > 0$ such that $\alpha < 1/8\, d(E_1^*, X-u)$ and $\alpha < \varepsilon/8$. Now, for each $d \in \hat{\mathcal{C}}$ such that $d \not\subset 0^*$, let $d^\# = (d-u) \cup (d \cap D_\alpha(d-u) \cap u) \cup (d \cap 0'^*)$. Each $d^\#$ is open in X and $\mathcal{C}^\# = \{d^\# | d \in \hat{\mathcal{C}}, d \not\subset 0^*\} \cup F$ is a taut open chain cover of X. Let $G = \{C^* | C$ is a maximal subchain of $\mathcal{C}^\#$ such that no link of C is in 0^*, but each link contains some point in X-u$\}$ and $H = \{C^* | C$ is a maximal subchain of $\mathcal{C}^\#$ such that no link of C is in 0^*, but each link is in u$\}$. Note that if c is a link of $\mathcal{C}^\#$, then exactly one of the following is true: A) $c \in F$, B) $c \in C$ for some $C^* \in G$, or

C) $c \in C$ for some $C^* \in H$. Then $\mathcal{C}'$, the taut open chain formed from members of $F \cup G \cup H$ covers X. We need to show that mesh $\mathcal{C}' < \varepsilon$.

Suppose $d \in G$. Then $d \not\subset u$. If $d \cap u = \phi$, diam $d < \varepsilon/8$, so suppose $d \cap u \neq \phi$. (*Case* 1.) Suppose d is an end link of $\mathcal{C}'$. Then d intersects exactly one link d' of $\mathcal{C}'$. Now $d' \notin G$, so either $d' \subseteq v$ or $d' \subseteq u$. If $d' \subseteq v$, diam $d < 3(\varepsilon/8)$, since $d \subseteq D_\alpha(X-u)$. (Otherwise, $d \cap 0'^* \neq \phi$, and $d \cap 0'^*$, $d \cap d'$ are non-empty separated open sets in d.) Then suppose $d' \subseteq u$. Some component Q of $\overline{d}$ intersects both $d-u$ and $d' \cap d$. (There is x in $d-u$, and Q is the component of $\overline{d}$ containing x, $Q \cap \partial d \neq \emptyset$. But $\partial d \subseteq d'$. Further, $(d-u) \cap Q$ and $\partial d \cap Q$ are subsets of $d - d'$ and $d' - d$, respectively. Then $Q \cap d \cap d' \neq \emptyset$, for otherwise Q is the union of 2 disjoint closed sets.) Then $d' \subseteq D_i$ for some $D \in \mathcal{D}$, $i = 0$ or m_D, and $\overline{D}_i \cap \partial u \neq \phi$. For convenience, let $i = 0$. If $d \cap 0'^* \neq \phi$, then $d \cap 0'^* \subseteq D_1$, for there is some point z in $d \cap 0'^*$, and if Q_z is the component of $\overline{d}$ which contains z, Q_z must intersect d'. Then $d \subseteq D_\alpha(X-u) \cup D_0 \cup D_1$, and diam $d < 5(\varepsilon/8) < \varepsilon$. (*Case* 2.) Suppose d is not an end link of $\mathcal{C}'$. Then d intersects 2 links d' and $\hat{d}$ of $\mathcal{C}'$. If $d \subseteq D_\alpha(X-u)$, diam $d < 3(\varepsilon/8)$, so suppose $d \not\subseteq D_\alpha(X-u)$. Then $d \cap 0'^* \neq \emptyset$, since by construction $d \subseteq D_\alpha(X-u) \cup 0'^*$. Neither d' nor d is in G. (i) Suppose both are subsets of v. Then $d \cup d' \cup \hat{d} \subseteq D_\alpha(X-u) \cup 0'^*$, with $d' \cup \hat{d} \subset D_\alpha(X-u)$ and $d \cap 0'^* \neq \emptyset$, which is an impossibility. At least one of d' and $\hat{d}$ is a subset of u, say d'. (ii) Suppose that $\hat{d} \subseteq v$. Then $d' \in F$, for otherwise $d' \in H$,

$d \cup d' \cup \hat{d} \subseteq D_\alpha(X-u) \cup 0'^*$, $\hat{d} \subset D_\alpha(X-u)$, and $d' \subset 0'^*$, an impossibility. Further $\overline{d}'$ must contain a boundary point of u, for otherwise $d \cup d' \cup \hat{d} \subseteq E_1^* \cup D_\alpha(X-u)$, with $\hat{d} \subseteq D_\alpha(X-u)$ and $d' \subset E_1^*$. Suppose D_k denotes that set in $\hat{\mathcal{D}}$ containing d'. It is also the case that $d \cap 0'^*$ is a subset of exactly one set in 0', and that set must be a link D_j adjacent to D_k, for otherwise $(d \cap 0'^*) - D_j$ and $d' \cup \hat{d} \cup (d \cap D_j) \cup D_\alpha(X-u)$ are mutually exclusive closed (in $d' \cup \hat{d} \cup d$) sets whose union is $d' \cup \hat{d} \cup d$. Then $d \subseteq D_\alpha(X-u) \cup D_k \cup D_j$ and diam $d < 5(\varepsilon/8)$. (iii) Suppose finally that $\hat{d} \not\subseteq v$. Then $\hat{d} \subseteq u$ and $d' \subseteq u$. If both $\hat{d}$ and d' are in H, then $d \cup d' \cup \hat{d} \subseteq D_\alpha(X-u) \cup 0'^*$, with $d' \cup \hat{d} \subseteq 0'^*$ and $d-u \neq \emptyset$, which is not possible. One of $\hat{d}$ and d', say d', is in F. Let D denote that chain of $\mathcal{D}$ such that $d' \subseteq D^*$, and let D_k denote that link of D that contains d'. Some component Q of $\overline{d}$ intersects both $d \cap d'$ and $d \cap \hat{d}$. However, $\hat{d} \subseteq D^*$, also. (Otherwise there is some subcontinuum Q' of Q such that $Q' \subseteq X-u$, $Q' \cap \partial u \cap \overline{D}'^* \neq \emptyset$, and $Q' \cap \partial u \cap \overline{D}^* \neq \emptyset$ for some $D' \neq D \in \mathcal{D}$, which is a contradiction to the choice of v.) If $\hat{d} \in H$, $\hat{d}$ must be in a link D_ℓ of D adjacent to D_k, for $\hat{d}$ cannot intersect any other link of 0', since this would mean that Q would intersect some other link of F besides D_k and that d intersects more than 2 links in the chain. Also D_k must contain a boundary point of u, since $d-u \neq \emptyset$ and $\hat{d} \cap \partial u = \emptyset$. Then $d \subseteq D_\alpha(X-u) \cup D_k \cup D_\ell$ and diam $d < 5(\varepsilon/8)$. On the other hand, if $\hat{d} \in F$, $\hat{d}$ must be in D_k also, for otherwise Q intersects $0'^* - F^*$ and d intersects more than 2 other links. Again, diam $d < \varepsilon$.

Suppose $d \in H$. Then $d \subseteq u$, and $d \subseteq 0'^*$. (*Case* 1.) If d is an end link of $\mathcal{C}'$, and intersects only the link d' of $\mathcal{C}'$, $d' \subseteq 0^*$, and there is $D_j \in 0$ such that $d' \subseteq D_j$. Since $d \cap d' \neq \phi$, d must intersect D_k where $D_k \cap D_j \neq \phi$. But then d is a subset of the union of a subchain of D containing at most 3 links, and diam $d < 3\varepsilon/8$. (*Case* 2.) If d is not an end link of $\mathcal{C}'$, then d intersects two links d' and $\hat{d}$ of $\mathcal{C}'$. Now d' and $\hat{d}$ are in 0^*, so there are $D_j \in \mathcal{D}^* \cap 0$ such that $d' \subseteq D_j$ and $D_k' \in \mathcal{D}^* \cap 0$ such that $\hat{d} \subseteq D_k'$. Then $D = D'$, since some component Q of $\bar{d}$ must intersect both $d \cap \hat{d}$ and $d \cap d'$. Also, $d \subseteq D^*$, and $|j - k| \leq 2$. Finally d can only intersect elements of $D \cap 0'$ which are adjacent to D_j and D_k, and d is a subset of the union of a subchain of D containing at most 5 links. Then diam $d < 5\varepsilon/8 < \varepsilon$.

Note that continua whose proper subcontinua are chainable <u>do</u> admit collections of the kind needed in Theorem 10. That is, suppose that X is a continuum each of whose proper subcontinua is chainable, and that $o = \{o_1,\cdots,o_n\}$ is a finite collection of open sets in X such that $\bar{o}_i \cap \bar{o}_j = \emptyset$ for $i \neq j$. There is an open set o_0 such that
(1) $o' = \{o_1,\cdots,o_{n_1},o_0\}$ is a finite collection of open sets in X such that $\bar{o}_i \cap \bar{o}_j = \emptyset$ for $i \neq j$, and (2) each component of $X - o'^*$ intersects the boundaries of no more than 2 elements of o'. (There is $\varepsilon > 0$ such that $\varepsilon < \min_{i \neq j} d(\bar{o}_i,\bar{o}_j)$. If $o_0 = \overline{X - D_{\varepsilon/2}(\bar{o}^*)}$, where $D_{\varepsilon/2}(\bar{o}^*) = \{y \in X | d(y,\bar{o}^*) < \varepsilon/2\}$,

then, each component of $X - (o \cup \{o_0\})^*$ intersects the boundary of no more than one element of o.)

Lemma 11 tells us that we can easily modify the collection o slightly and get the resulting collection o' to have even nicer properties. That is, if $\varepsilon > 0$ and $o = \{o_1,\cdots,o_n\}$ is a finite collection of open sets in X such that $\bar{o}_i \cap \bar{o}_j = \emptyset$ for $i \neq j$, then there is an open set o_i' such that (1) $\bar{o}_i \subset o_i' \subseteq D_\varepsilon(\bar{o}_i)$ for each i, (2) $\bar{o}_i' \cap \bar{o}_j' = \emptyset$ for $i \neq j$, and (3) for each i there is a finite collection $\mathcal{C}_i$ of mutually exclusive taut open (in $\overline{X - \cup o_i'}$) chains such that properties (1)-(4) of Lemma 6 are satisfied. Then, by Lemma 11, each component of $\overline{X - \cup o_i'} = \bar{u}$ intersects ∂u in no more than 2 points, each of those points is an end point of the component, and if the component does intersect ∂o in 2 end points, those points are opposite end points of the component. We need to define what we mean by an end point of a chainable continuum. If C is a chainable continuum and $x \in C$, x is an *end point* of C if there exists a definint sequence $C_1,C_2,\cdots$ of open chain covers of C such that x is in an end link of C_i for each i. Further x and y are *opposite end points* of C if there exists a defining sequence $C_1,C_2,\cdots$ of open chain covers of C such that x is in one end link of C_i and y is in the other for each i [1].

Lemma 11. *Suppose that* X *is a continuum with the property that every proper subcontinuum is chainable. Suppose that* o *is an open subset of* X *such that* $\bar{o} \neq X$. *Then the following are equivalent:*

(1) *Each component of* $\overline{o}$ *intersects* ∂o *in no more than 2 points, each of those points is an end point of the component, and if a component does intersect* ∂o *in 2 points then those points are opposite end points of the component.*

(2) *There is a sequence* $\mathcal{C}_1, \mathcal{C}_2, \cdots$ *of finite collections of mutually exclusive taut open (in* $\overline{o}$*) chains such that* $\mathcal{C}_1, \mathcal{C}_2, \cdots$ *satisfies properties* 1-4 *of the statement of Lemma* 6.

Proof. ((2) $\Rightarrow$ (1)). Property 4 of Lemma 6 gives this result ((1) $\Rightarrow$ (2)). Suppose $\varepsilon > 0$. Suppose that for each component Q of $\overline{o}$, C_Q is a taut chain covering Q essentially such that (1) mesh $C_Q < \varepsilon$, (2) $\partial o \cap C_Q^*$ is contained in the end links of C_Q, and (3) no point of $\partial o \cap C_Q^*$ is in the closure of a non-end link of C_Q. Consider the decomposition of $\overline{o}$ into its components, and let E denote the decomposition space, which is totally disconnected, and let P denote the projection map from $\overline{o}$ onto its decomposition E. For each Q in E, $Q \subseteq \overline{o}$ and not only is it true that $P(C_Q^*)$ contains Q in its interior, but also it is true that $Q \notin \overline{P(\overline{o} - C_Q^*)}$: Otherwise $Q \in \overline{P(\overline{o} - C_Q^*)}$ and there is a sequence $Q_1, Q_2, \cdots$ in $P(\overline{o} - C_Q^*)$ which converges to Q in E. Then Q_i, considered as a set in $\overline{o}$, is not a subset of C_Q^*. Thus there is $x_i \in Q_i - C_Q^*$, and we may assume without loss of generality that $x_1, x_2, \cdots$ converges to x in Q. But then $x \in C_Q^*$, and this can't be. Choose u_Q open in E such that $Q \in u_Q = \overline{u}_Q \subseteq P(C_Q^*)$, $u_Q \cap \overline{P(\overline{o} - C_Q^*)} = \emptyset$. Then $Q \subseteq P^{-1}(u_Q) \subseteq C_Q^*$. Let $\hat{C}_Q$ denote that chain covering Q essentially whose links are the links of C_Q intersected with $P^{-1}(u_Q)$. The chain

$\hat{C}_Q$ has the property that its union is both open and closed in $\bar{o}$, and, thus, that each component of $\bar{o}$ is either contained in $\hat{C}^*_Q$ or does not intersect $\hat{C}^*_Q$ at all.

Now $\mathcal{C} = \{\hat{C}^*_Q | Q$ is a component of $\bar{o}\}$ is an open cover of $\bar{o}$ and some finite subcollection $\mathcal{C}'$ of $\mathcal{C}$ covers $\bar{o}$. Suppose that $\mathcal{C}' = \{C_1, C_2, \cdots C_n\}$ denotes the finite subcollection of $\mathcal{C}$ that covers $\bar{o}$. Let $D_1 = C_1$, $D_i = C_i - \cup_{j<i} C_j$ for $i > 1$. Without loss of generality, assume $D_i \neq \emptyset$ for any i. Let $F = \{c \cap D_i | 1 \leq i \leq n$ and c is a link in some chain whose union is in $\mathcal{C}'\}$. Then F is collection of open sets in $\bar{o}$ and if we let $\hat{F} = \{E | E$ is a maximal chain formed of sets in $F\}$, then $\hat{F}$ is a collection of taut chains in $\bar{o}$ such that (1) mesh $C < \varepsilon$ for each chain C in $\hat{F}$, (2) $\partial o \subset \{\partial A | A$ is an end link of a chain in $\hat{F}\}$, (3) each point of ∂o is in the closure of exactly one link of C. It is possible that some chain $E \in \hat{F}$ does not cover any continuum essentially, but we can modify E easily to change this. Let $A = \{Q | Q$ is a component of E^* and $Q \cap E_o \cap \partial o \neq \emptyset\}$ and $B = \{Q | Q$ is a component of E^* and $Q \cap E_m \cap \partial o \neq \emptyset\}$ where E_o and E_m represent the 2 end links of E. Then A^* and B^* are mutually exclusive closed sets in $\bar{o}$ (assuming E does not cover any continuum essentially) and we replace E in $\hat{F}$ with $E_A = \{e \cap A^* | e$ is a link in $E\}$ and $E_B = \{e \cap B^* | e$ is a link in $E\}$. Further, we may also assume that each chain E in $\hat{F}$ contains 5 links for if the chain E has fewer than 5 links, we can again modify it, using Lemma 4, so that it does. The resulting collection G of taut open (in $\bar{o}$) chains has the following properties: (1) mesh $G < \varepsilon$, and $G^{**} = \bar{o}$, (2) each chain in G essentially covers a continuum and contains at least 5 links, and (3) $\partial o \subseteq \{\partial A | A$ is an end link of a chain in G and each point of ∂o is in the closure of exactly one link of $G^*\}$. Using the techniques of the proof of Lemma 6, we can then construct the desired sequence of chain covers of $\bar{o}$.

Corollary 12. *Suppose that* X *is a continuum with the property that every proper subcontinuum is chainable. Suppose* X *has the property that whenever* o *is an open set in* X *such that* $\bar{o} \neq X$ *and such that each component of* $\bar{o}$ *intersects the boundary in no more than* 2 *points, each of those points is an end point of the component, and if a component intersects the boundary in* 2 *points, then those points are opposite end points of the component, then there is a taut (circular) chain cover* $\mathcal{C}$ *of* X *such that* o *is a union of links of* $\mathcal{C}$. *Then* X *is (circularly) chainable.*

Proof. Essentially the proof of Theorem 10 is used here, when we note that the open set u of that proof can be chosen as one of the components-don't-intersect-the-boundary-more-than-twice sets, as can each member of the collection $\mathcal{O}$ there, because of Lemmas 6 and 11.

We end with a question:

Question. Is the converse of Corollary 12 true? In other words, do the above conditions characterize (circular) chainability?

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