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REALIZING WHITEHEAD TORSION BY SELF-EQUIVALENCES ON 2-COMPLEXES WHEN $\pi_1 = Q_n$

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WHEN $\pi_1 = \mathbf{Q}_n$**

M. Paul Latiolais¹

0. Introduction

In the study of Simple Homotopy Theory, the general question one asks is: Given a homotopy equivalence $f: K \rightarrow L$, when does there exist a simple homotopy equivalence $g: K \rightarrow L$? (i.e. When does K deform to L ?). See Cohen [C], Chapter 2, for definitions.

In general, homotopy type and simple homotopy type are different. However, it is not known whether or not homotopy equivalent 2-complexes are simple homotopy equivalent. This problem is directly related to a problem in Combinatorial Group Theory.

If we assume that each of our finite 2-complexes have a single 0-cell, then we may relate to it a unique presentation of its fundamental group. And given any finite presentation of a group, there exists a unique finite 2-complex related to it (see [W] and [M1]). Since all finite complexes deform to ones with a single 0-cell by collapsing a maximal tree (see [M1]), we will assume that our complexes have only one 0-cell.

Consider the question of whether or not two 2-complexes K^2 and L^2 deform to one another with deformations of at most dimension 3. That is equivalent to the question of

¹A major portion of the research for this article was done while the author was at Tulane University.

whether or not their related group presentations can be transformed to one another by Q^{**} -transformations (see [M1] for definitions and [W] for proof).

In general, if two n -complexes deform to one another, then one can be deformed to the other via deformations of dimension at most $n + 1$, $n > 2$. It is still an unknown question whether or not simple homotopy equivalent 2-complexes always 3-deform to one another.

The question of whether or not two 2-complexes are simple homotopy equivalent is equivalent to the question of whether or not their related group presentations deform to one another via "generalized" Q^{**} -transformations. See R. Brown [B] for definitions and proof.

Related to the question of homotopy equivalence versus simple homotopy equivalence is the study of Whitehead torsion of self-equivalences (see [C], (24.4) and [M2]). The relation is given by the following fact noted by Cockcroft and Moss in [C-M], Corollary 2.

0.1 Proposition. Given any finite CW-complex K , then every element of $Wh(K)$ is realizable as the torsion of a homotopy equivalence from K to K if and only if any finite CW-complex L homotopy equivalent to K is simple homotopy equivalent to K .

Consequently, it is important and interesting to know something about the elements of $Wh(K)$ which are realizable as torsions of self-equivalences on K . The importance of (0.1) is exhibited in the following theorem and corollary

of [L1], Theorem 5.3 and Theorem 5.4 (also see [L2], Theorems 2.2 and 2.3).

0.2 Theorem. If K is a finite 2-dimensional CW-complex with $\pi_1(K)$ finite and abelian, then every element of $\text{Wh}(K)$ is realizable as the torsion of a self-equivalence of K .

0.3 Corollary. Given any finite 2-complex K with $\pi_1(K)$ finite and abelian, then any finite 2-complex L homotopy equivalent to K is simple homotopy equivalent to K (i.e. homotopy type and simple homotopy type are equivalent for finite complexes with finite abelian fundamental group).

W. Metzler in [M2] gives an example of a finite 2-complex with infinite fundamental group for which the above theorem is not true. Our motivation was to find a counter-example to (0.2) where finite abelian fundamental group is replaced by finite non-abelian fundamental group.

We indeed found a counter-example, but not necessarily the one we were looking for. In our example, the self-equivalences which induce the identity map on the fundamental group do not represent all torsion. This example is therefore either a counter-example to (0.2) in the non-abelian case or it is the first known example of a 2-complex in which the realizable torsion of a self-equivalence depends on the induced map on π_1 .

After describing our counter-example, we will generalize the results of our example to show that the units of $\text{Wh}(K)$ are always realizable when $\pi_2(K)$ is singly generated as a $\mathbb{Z}\pi_1$ -module.

In the last section of this paper, we will discuss partial results and failures to determining whether or not our example is a counter-example to (0.2) in the non-abelian case.

1. Properties of a "Counter-Example"

We are looking for a finite 2-complex K with finite non-abelian fundamental group such that there exists an element of $\text{Wh}(K)$ which is not the torsion of any self-homotopy equivalence on K . There are some obvious things we should look for. We want a 2-complex with:

(1) $\text{Wh}(K) \neq 0$. The reason is obvious.

(2) $\text{Wh}(K)$ should contain more than units ("units" meaning elements representable as 1×1 matrices). In all known examples, including Metzler's counter-example [M2], the units are all realizable. It seems reasonable to conjecture that units are always realizable.

(3) $\chi(K)$, the Euler characteristic of K , should be minimal with respect to $\pi_1(K)$. $\chi(K)$ being minimal makes it more likely that non-realizable elements of $\text{Wh}(K)$ will occur. If, for example, $\chi(K)$ were two above the minimum, then all of the elements of $\text{Wh}(K)$ would automatically occur, see [D1], Theorem 2.

(4) We need to know something about the homotopy tree of $(G, 2)$ -complexes, $G = \pi_1(K)$. That is, what are the homotopy classes of finite 2-complexes with fundamental group G (see [D1] and [D-S]). If we know something about the homotopy tree, we may be able to say something about the simple homotopy tree. See (2.5).

Notice that criteria (1) and (2) are really stipulations about the group $G = \pi_1(K)$, where K is the standard complex of the presentation of G of maximum deficiency, this presentation being guaranteed by criterion (3).

(3) and (4) are stipulations about what is known about groups which satisfy (1) and (2), e.g. it is not always known whether or not a given presentation of a group has maximum deficiency.

As an example, let us consider $G = D_n$, the dihedral group of order $2n$, n odd. (1) is satisfied. $\text{Wh}(ZD_n) \neq 0$, by Jajodia and Magurn [J-M], Theorem 6 and Theorem 10 and Note 7. However, (2) is not satisfied. Magurn showed that $\text{Wh}(ZD_n)$ contained only units. As for (3), D_n has a presentation of deficiency zero,

$$P(D_n) = \{a, b \mid a^n b^{-2}, [b, a^{(m-1)/2}] a^{-1}\}.$$

Consequently, the standard complex $K(P)$ of the presentation will have minimal Euler characteristic. The homotopy tree of $(D_n, 2)$ -complexes is a stalk, by Dyer [D1], Example 3, p. 223. That is, all finite 2-complexes with the fundamental group D_n and the same Euler characteristic are homotopy equivalent.

And lastly, Jajodia and Magurn ([J-M], Theorem 12) were able to show that the units of $\text{Wh}(K(P))$ were all realizable as the torsions of self-equivalences on $K(P)$. Therefore, every element of $\text{Wh}(K)$ is realizable as the torsion of a self-equivalence on K , for any finite 2-complex K with fundamental group D_n .

2. The Counter-Example

Let $Q_n = \{a, b | a^n b^{-2}, abab^{-1}\}$ be the dicyclic group of order $4n$. If $n = 2$, Q_n is the quaternions group. If $n = 2^i$, Q_n is known as the generalized quaternions group.

So how does Q_n stack up under our criteria for a counter-example?

(1) $Wh(ZQ_n) \neq 0$, for at least some n , by Higman [H] Theorem 11.

(2) $Wh(ZQ_n)$ is not just units, for some n , by Magurn, Oliver and Vaserstein [M-O-V], Theorem 7.16 and Theorem 7.18.

(3) Q_n has a presentation of deficiency zero, i.e. the one above.

(4) The homotopy tree for Q_n has height ≤ 1 . (See [D1], p. 224.) That is, there may be complexes with minimal Euler characteristic of different homotopy type, but above the minimum Euler characteristic, any two finite 2-complexes with fundamental group Q_n and the same Euler characteristic will be homotopy equivalent.

2.1 Theorem. Let K be the standard complex of the presentation $\{a, b | a^n b^{-1}, abab^{-1}\}$ of Q_n (the dicyclic group of order $4n$), and $f: K \rightarrow K$ a homotopy equivalence with $f_{\#}: \pi_1(K) \rightarrow \pi_1(K)$ the identity map, then $\tau(f) \in Wh(K)$ is a unit. All of the units of $Wh(K)$ are realizable in this way.

Proof. The complex K above is the complex with a single 0-cell e_0 . K has two 1-cells attached to e_0 on their boundaries. We will call these 1-cells "a" and "b." We orient a and b, so we can talk about a^{-1} and b^{-1} as a and b, but in the opposite direction. We are purposely

confusing these 1-cells with the generators of Q_n . K also has two 2-cells, which we will call R_1 and R_2 . The 2-cells are attached to the 1-cells on their boundary so that $\partial R_1 = a^n b^{-2}$ and $\partial R_2 = abab^{-1}$.

We will first show that all units are realizable. Suppose $f: K \rightarrow K$ is a homotopy equivalence which induces the identity map on $\pi_1(K)$. Let \tilde{K} be the universal cover of K . Since $f_\#: \pi_1 \rightarrow \pi_1$ is the identity, then f is homotopic to a map which induces the identity on $C_1(\tilde{K})$. Since we are only interested in homotopy classes of maps, we may assume f is that map. Now f induces the following map on the chain complex of the universal cover \tilde{K} :

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 H_2(\tilde{K}) & \xrightarrow{\tilde{f}_2^*} & H_2(\tilde{K}) \\
 \partial \downarrow & & \partial \downarrow \\
 C_2(\tilde{K}) & \xrightarrow{\tilde{f}_2} & C_2(\tilde{K}) \\
 \partial \downarrow & & \partial \downarrow \\
 C_1(\tilde{K}) & \xrightarrow{\tilde{f}_1 = \text{id}} & C_1(\tilde{K}) \\
 \partial \downarrow & & \partial \downarrow \\
 C_0(\tilde{K}) & \xrightarrow{\tilde{f}_0 = \text{id}} & C_0(\tilde{K}) \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}
 \tag{1}$$

2.2 Lemma. Given a finite 2-dimensional CW-complex K and a $\mathbb{Z}\pi_1(K)$ -module map $\phi: C_2(\tilde{K}) \rightarrow C_2(\tilde{K})$ which commutes with the boundary operator, then there exists a homotopy equivalence $f: K \rightarrow K$ such that $\tilde{f}_1 = \text{id}$ and $\tilde{f}_2 = \phi$ if and only if the $\mathbb{Z}\pi_1$ -matrix representative of ϕ , M , is invertible. The resulting homotopy equivalence f will induce the identity map on $\pi_1(K)$, and $\tau(f) = [M] \in \text{Wh}(K)$.

Proof. See [L2], Lemma 1.4.

Proof of (2.1), continued 1. Let M_f be the $Z\pi_1(K)$ -matrix representative of \tilde{f}_2 . By (2.2), $M_f \in GL_2(ZQ_n)$. Also by (2.2), any element of $GL_2(ZQ_n)$ which commutes with ∂_2 will represent the map on $C_2(\tilde{K})$ of some self-homotopy equivalence. Any such matrix must be of the form:

$$(3) \quad \text{from} \quad \begin{matrix} & \text{to} \\ & \begin{matrix} \tilde{R}_1 & \tilde{R}_2 \end{matrix} \\ \begin{matrix} \tilde{R}_1 \\ \tilde{R}_2 \end{matrix} & \left(\begin{array}{cc} 1 + \phi_1 & \phi_2 \\ p_1 & 1 + p_2 \end{array} \right) , \end{matrix}$$

where \tilde{R}_1, \tilde{R}_2 are the preferred lifts of the two 2-cells of K , and $\phi_1\tilde{R}_1 + \phi_2\tilde{R}_2$ and $p_1\tilde{R}_1 + p_2\tilde{R}_2$ are elements of $H_2(\tilde{K})$. The above statement is proved using diagram (3a), with $\phi = \tilde{f}_2$.

$$(3a) \quad \begin{array}{ccc} C_2(\tilde{K}) & \xrightarrow{\phi - \text{id}} & C_2(\tilde{K}) \\ \partial \downarrow & & \partial \downarrow \\ C_1(\tilde{K}) & \xrightarrow{0} & C_1(\tilde{K}) \end{array}$$

Using Fox derivatives (see [F]) it is easy to show that $(a-1)\tilde{R}_1 + (1-ab)\tilde{R}_2$ is an element of $H_2(\tilde{K})$. In fact, $(a-1)\tilde{R}_1 + (1-ab)\tilde{R}_2$ generates $H_2(\tilde{K})$ (see (2.4)). So the following matrix is of the form (3), with $p_1 = p_2 = 0$:

$$(4) \quad \begin{matrix} & \tilde{R}_1 & \tilde{R}_2 \\ \begin{matrix} \tilde{R}_1 \\ \tilde{R}_2 \end{matrix} & \left(\begin{array}{cc} 1 + \delta(a-1) & \delta(1-ab) \\ 0 & 1 \end{array} \right) \end{matrix}$$

which is Whitehead equivalent to $[1 + \delta(a-1)]$.

2.3 Lemma. *All of the units of $Wh(ZQ_n)$ may be represented in the form $1 + \delta(a-1)$, where $\delta \in ZQ_n$ and "a" is the generator in the presentation $[a, b | a^n b^{-2}, abab^{-1}]$.*

Proof. Note first that in this proof we will freely use the facts $a^{2n} = 1$, $b^4 = 1$, $ab = ba^{-1}$, $a^{-1}b = ba$, etc.

Let $\theta \in ZQ_n$, then $\theta = \sum_i m_i \gamma_i$, where $m_i \in Z$ and $\gamma_i \in Q_n$, for each i , and i is finite.

Consider the map $A: ZQ_n \rightarrow Z(Z_2)$, which sends $a \mapsto 1$ and $b \mapsto b$ (We are considering Z_2 in multiplicative notation with generator b .) If θ is a unit then $A(\theta) = \alpha$ is a unit in $Z(Z_2)$. But the units in $Z(Z_2)$ are all trivial (see [C], (11.5)), so $A(\theta) = \alpha = \pm b^r$, $r = \pm 1$ or 0 . But $[(\pm b^r)^{-1}] \theta$ represents the same element as θ in $Wh(ZQ_n)$. So we may assume without loss of generality that $A(\theta) = 1$.

Now consider $p = 1 - \theta$. $A(p) = 0$. Since all elements of Q_n may be expressed as $b^\epsilon a^k$; with $\epsilon = 0$ or 1 and $k = 0, \dots, 2n - 1$, then p may be expressed as $p = \sum_i n_i b^{\epsilon_i} a^{k_i}$.

Since $A(p) = 0$, then $\sum_i n_i b^{\epsilon_i} = 0$. Consequently,

$$\begin{aligned} p &= \sum_i n_i b^{\epsilon_i} a^{k_i} \\ &= \sum_i n_i b^{\epsilon_i} a^{k_i} - \sum_i n_i b^{\epsilon_i} \\ &= \sum_i n_i b^{\epsilon_i} (a^{k_i} - 1). \end{aligned}$$

Since $(a - 1)$ is a factor of $(a^{k_i} - 1)$, we have $p = \sum \phi_i (a - 1) = (\sum \phi_i) (a - 1) = -\delta (a - 1)$, with $-\delta = \sum \phi_i$, and therefore $\theta = 1 - p = 1 + \delta (a - 1)$.

So, by the above lemma, any unit of $Wh(K)$ is representable as $1 + \delta (a - 1)$, which is Whitehead equivalent to

$$\begin{bmatrix} 1 + \delta (a - 1) & \delta (1 - ab) \\ 0 & 1 \end{bmatrix},$$

which is invertible, since $1 + \delta (a - 1)$ is a unit.

Therefore by (2.2), since M is of form (3), every unit of

$\text{Wh}(K)$ is realizable as the torsion of a self-equivalence f on K , with $f_{\#} = \text{id}$.

To finish the proof of (2.1) we need to show that if f is a self-equivalence on K with $f_{\#} = \text{id}$, then $\tau(f)$ must be a unit. To do this we need one last lemma.

2.4 Lemma. Let K be the standard complex of the presentation $[a, b | a^n b^{-1}, abab^{-1}]$ of Q_n and let \tilde{R}_1 and \tilde{R}_2 be the preferred lifts in the universal cover \tilde{K} of the two 2-cells of K (\tilde{R}_1 and \tilde{R}_2 generate $C_2(\tilde{K})$ as a $\mathbb{Z}[\pi_1(K)]$ -module). Then $(a-1)\tilde{R}_1 + (1-ab)\tilde{R}_2$ generates $H_2(\tilde{K})$.

Proof. Only an outline will be given. The proof of this lemma is trivial, but tedious and very lengthy. All one needs to do is to follow the example of [M2], p. 330. Let $\xi = (\sum_i n_i a^{m_i} b^{\epsilon_i})\tilde{R}_1 + (\sum_i n'_i a^{m'_i} b^{\epsilon'_i})\tilde{R}_2$ be an element of $H_2(\tilde{K})$. We know that $(a-1)\tilde{R}_1 + (1-ab)\tilde{R}_2$ is an element of $H_2(\tilde{K})$, so we may subtract multiples of it to simplify the \tilde{R}_1 coefficient. Then take the boundary, noting that all of the coefficients must be zero.

Proof of (2.1) continued 3. Using matrix (3), by (2.4), the torsion of a self-equivalence which induces the identity on π_1 can always be represented in the form,

$$(5) \quad \begin{bmatrix} 1 + \delta_1(a-1) & \delta_1(1-ab) \\ \delta_2(a-1) & 1 + \delta_2(1-ab) \end{bmatrix},$$

So all we need to show is that given any invertible matrix of the form (5) is Whitehead equivalent to a unit.

$$\begin{pmatrix} 1+\delta_1(a-1) & \delta_1(1-ab) \\ \delta_2(a-1) & 1+\delta_2(1-ab) \end{pmatrix} \sim \begin{pmatrix} 1+\delta_1(a-1) & \delta_1(1-ab) & 0 \\ \delta_2(a-1) & 1+\delta_2(1-ab) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1+\delta_1(a-1) & \delta_1(1-ab) & 0 \\ \delta_2(a-1) & 1+\delta_2(1-ab) & 0 \\ -(a-1) & -(1-ab) & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \delta_1 \\ 0 & 1 & \delta_2 \\ -(a-1) & -(1-ab) & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & \delta_1 \\ 0 & 1 & \delta_2 \\ 0 & 0 & 1+(a-1)\delta_1+(1-ab)\delta_2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+(a-1)\delta_1+(1-ab)\delta_2 \end{pmatrix}$$

$\sim (1+(a-1)\delta_1+(1-ab)\delta_2)$, which therefore must be a unit.

2.5 Corollary. If L is a finite 2-complex with $\pi_1(L)$ equal to the generalized quaternions $(Q_n, n = 2^i)$ for some i and $\chi(L) > 1$, then any finite 2-complex homotopy equivalent to L is simple homotopy equivalent to L . All such 2-complexes are simple homotopy equivalent to $Kv_n S^2$ (K with n S^2 's attached), where K is the complex of (2.1) and $n = \chi(L) - 1$.

Proof. The minimum possible Euler characteristic is 1. So by Dyer [D1] Example 4, p. 223, all complexes with fundamental group Q_n and the same Euler characteristic above 1 will be homotopy equivalent. In particular, they

are all homotopy equivalent to $Kv_n S^2$, where n is the Euler characteristic minus 1. Now since K realizes all of its torsion, by [M-O-V], then K is simple homotopy equivalent to any complex homotopy equivalent to it. But since K realizes all of its Whitehead torsion, so will $Kv_n S^2$, and the same follows.

So what about a counter-example?

2.6 Corollary. Let K be the standard complex of the presentation $[a, b | a^n b^{-1}, abab^{-1}]$ of Q_n , where either $n = p$ is prime and the class number h_p is even, or $n = 4p$ with p prime and $p \equiv -1 \pmod{8}$. Then either

(1) not every element of $Wh(K)$ is realizable as the torsion of a self-equivalence, or

(2) there exists an automorphism ϕ of $\pi_1(K) = Q_n$, $\phi \neq$ identity, and there exists a self-equivalence f which induces $f_{\#} = \phi$, such that the torsion of f is not a unit or zero. I.e., any self-equivalence which induces ϕ will have non-unit, non-zero torsion.

Proof. By [M-O-V], $Wh(K)$ contains non-units.

The author would prefer that (1) were true, but (2) is of definite interest, since it would be the first example in dimension 2 of different automorphisms of π_1 giving distinct torsions.

When the author was giving details of the proof of (2.1) at the impromptu-mini-pseudo-pre-Alta-conference at the University of Oregon, it was noted by Wolfgang Metzler that the matrix manipulation in the proof of (2.1) could be trivially generalized to give:

2.7 Theorem. Given any finite 2-complex K with $\pi_2(K)$ singly generated as a $\mathbb{Z}\pi_1(K)$ -module, then any self-equivalence on K which induces the identity map on $\pi_1(K)$ must have unit Whitehead torsion.

In particular, the above hypothesis is satisfied by the standard presentation of any 1-relator group presentation.

3. Realizable Automorphisms of $\pi_1(K)$

To show that K is a counter-example of sort (1) of (2.6), we need to show that given an automorphism ϕ of Q_n , then either there does not exist a self-equivalence that induces ϕ , or if there does exist a self-equivalence that induces ϕ , then there exists a simple self-equivalence that induces ϕ (simple \equiv zero torsion). This section will state partial results to the above question, and the obstructions to completing these results.

Now $\text{Aut}(Q_n)$ is of the following form. For any $\phi \in \text{Aut}(Q_n)$,

$$\phi(a) = a^\alpha, \text{ where } (\alpha, 2n) = 1, \text{ and}$$

$$\phi(b) = a^i b, \text{ for } 0 \leq i < 2n.$$

So the $\text{Aut}(Q_n)$ is generated by η_α and μ defined as follows. $\eta_\alpha(a) = a^\alpha$ and $\eta_\alpha(b) = b$. $\mu(a) = a$ and $\mu(b) = ab$. If there exists a number σ that is primitive with respect to $2n$, then $\text{Aut}(Q_n)$ has only two generators, η_σ and μ . Otherwise, η_α refers to several possible automorphisms, η_α , with $(\alpha, 2n) = 1$.

With respect to simple self-equivalences, μ is no problem. Using a lemma similar to (2.2) (see [L1], Theorem

1.4), we can construct a self-equivalence g , with $g_{\#} = \mu$ and $\tau(g) = 0$. The details will not be given here, since these are partial results, and possibly of no use.

As for η_{α} , the story is quite different. There are maps on K which can be constructed such that the induced map on $\pi_1(K)$ is η_{α} , but so far, all maps that have been constructed were not homotopy equivalences. The author would like to show that no homotopy equivalence exists which induces η_{α} , for any α relatively prime to $2n$, n satisfying (2.6). This would finish the proof of (2.6) (1).

We need to show that any map $f: K \rightarrow K$ that induces $f_{\#} = \eta_{\alpha}$ on $\pi_1(K)$ does not induce an isomorphism on $\pi_2(K) \cong H_2(\tilde{K})$. Let $\Gamma = (a-1)\tilde{R}_1 + (1-ab)\tilde{R}_2$ be our generator of $H_2(\tilde{K})$ from (2.4). Since \tilde{K} is 2-dimensional, $H_2(\tilde{K})$ is contained in $C_2(\tilde{K})$. So Γ can be thought of as an element of $C_2(\tilde{K})$. And $H_2(\tilde{K})$ can be thought of as the subgroup generated by Γ . As a singly generated $Z\pi_1$ -module, $H_2(\tilde{K}) \cong ZQ_n / \{\psi \mid \psi\Gamma = 0\}$. But note that $\psi[(a-1)\tilde{R}_1 + (1-ab)\tilde{R}_2] = 0$ if and only if $\psi = pN$, where $p \in Z(Q_n)$ is arbitrary and

$$N = \sum_{i=0}^{2n-1} a^i + \sum_{i=0}^3 a^i b.$$

Consequently, $H_2(\tilde{K}) \cong ZQ_n / \langle N \rangle$.

Given any map $f: K \rightarrow K$, not necessarily a homotopy equivalence, then \tilde{f}_2 sends $\Gamma \rightarrow \gamma_f \Gamma$, for some $\gamma_f \in ZQ_n$. Let $A: ZQ \rightarrow Z$ be the augmentation map (sum of the coefficients). Given $\eta_{\alpha} \in \text{Aut}Q_n$ and $\gamma \in ZQ_n$, there exists a map f with $f_{\#} = \eta_{\alpha}$ and $\gamma_f = \gamma$ if and only if $A(\gamma) = \alpha^2$.

We would like to show that $\tilde{f}_2^*: H_2(\tilde{K}) \rightarrow H_2(\tilde{K})$ is not an isomorphism. To do that we need to show that

$[\gamma_f] \in ZQ_n/\langle N \rangle$ is not a unit. γ_f is obviously not a unit (in most cases), since $A(\gamma_f) \neq \pm 1$.

3.1 *Fact.* Given $\eta_\alpha \in \text{Aut}Q_n$, $0 < \alpha < 2n$, $(\alpha, 2n) = 1$ then there exists a self-equivalence $f: K \rightarrow K$ with $f_\# = \eta_\alpha$ if and only if there exists $\gamma \in ZQ_n$ with $A(\gamma) = \alpha^2$ and $[\gamma] \in ZQ_n/\langle N \rangle$ is a unit.

There are other techniques to approach the problem of the existence of a homotopy equivalence. If one of these techniques bear fruit, we will be able to state a result about the units of $ZQ_n/\langle N \rangle$.

References

- [B] R. Brown, *Generalized group presentations and formal deformations*, Ph.D. Thesis, Illinois (1984).
- [C-M] W. H. Cockroft and R. M. F. Moss, *On the simple homotopy type of certain two-dimensional complexes*, J. London Math. Soc. (2), 5 (1972), 726-728.
- [C] M. Cohen, *A course in simple-homotopy theory*, Springer-Verlag (1970).
- [D1] M. Dyer, *On the essential height of homotopy trees with finite fundamental group*, Compositia Math. 36 (2) (1978), 209-224.
- [D-S] _____ and A. Sieradski, *Trees of homotopy types of two-dimensional CW-complexes*, Comm. Math. Helv. 48 (1973), 31-44.
- [F] R. H. Fox, *Free differential calculus I*, Ann. of Math. (1953), 547-560.
- [H] G. Higman, *The units of group-rings*, Proc. London Math. Soc. 46 (1940), 231-248.
- [J-M] S. Jajodia and B. Magurn, *Surjective stability of units and simple homotopy type*, J. Pure Appl. Algebra 18 (1980), 45-58.

- [L1] M. P. Latiolais, *Homotopy type versus simple homotopy type in dimension two*, Ph.D. Thesis, Tulane (1984).
- [L2] _____, *The simple homotopy type of finite 2-complexes with finite abelian fundamental group*, Trans. A.M.S. (to appear).
- [M-O-V] Magurn, Oliver and Vaserstein, *Units in Whitehead groups*, J. of Algebra 84 (2) (1983), 324-360.
- [M1] W. Metzler, *Über den Homotopietyp zweidimensionaler CW-Komplexe und Elementartransformationen bei Darstellungen von Gruppen durch Erzeugende und definierende Relationen*, J. Rieme Angew. Math. 285 (1976), 7-23.
- [M2] _____, *Two-dimensional complexes with torsion values not realizable by self-equivalences*, Homological Group Theory, L.M.S. Lecture Notes Series 36 (1977), 327-337.
- [P] D. Puppe, *Homotopiemengen und Ihre Induzierten Abbildungen I*, Math. Zeit. 69 (1958), 299-344.
- [W] P. Wright, *Group presentations and formal deformations*, Trans. A.M.S. 208 (1975), 161-169.

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